COURSE: PHYSICS (H)

SEMESTER: VI

PHY-CC-13.T: ELECTROMAGNETIC THEORY

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Video Links of some topics

1. Propagation of Electromagnetic Waves in free space-I
   https://youtu.be/ttvDpSD1mrs

2. Propagation of Electromagnetic Waves in Free Space-II
   https://youtu.be/HQvCiVL11ds

3. Propagation of Electromagnetic Waves in Isotropic Dielectric medium
   https://youtu.be/0mDFVZeH290

4. Propagation of Electromagnetic Waves in a Conducting Medium-I
   https://youtu.be/Ri8eeIfa5wE

5. Propagation of Electromagnetic Waves in a Conducting Medium-II
   https://youtu.be/SXFfm59HFvE

Laws of Electromagnetic induction.

There are two laws of electromagnetic induction formulated by Faraday. The first law deals with the phenomenon and the second law gives the magnitude of induced emf. And third law is given by Lenz called Lenz's law which gives the direction of induced current

Faraday's Law of Electromagnetic induction.

From the experimental observations, Faraday enunciated two laws called Faraday's laws.

1st law: Whenever there is change of magnetic flux linked with a closed circuit, an emf is induced in the circuit and this induced emf lasts so long as the change in magnetic flux actually takes place.

2nd law: The magnitude of induced emf is directly proportional to the rate of change of magnetic flux linked with the circuit.

If $d\phi$ is the change in magnetic flux in dt seconds, then the induced emf is given by

$$\mathcal{E} \propto \frac{d\phi}{dt} \quad \text{or} \quad \mathcal{E} = -K\frac{d\phi}{dt}$$

Where K is the constant of proportionality and in SI, $K = 1$. Negative sign shows that the induced emf opposes the change in magnetic flux.

$$\mathcal{E} = -\frac{d\phi}{dt}$$

Since $\phi = \iint_S \vec{B} \cdot \vec{dS}$

$$\mathcal{E} = -\frac{d}{dt}\iint_S \vec{B} \cdot \vec{dS}$$

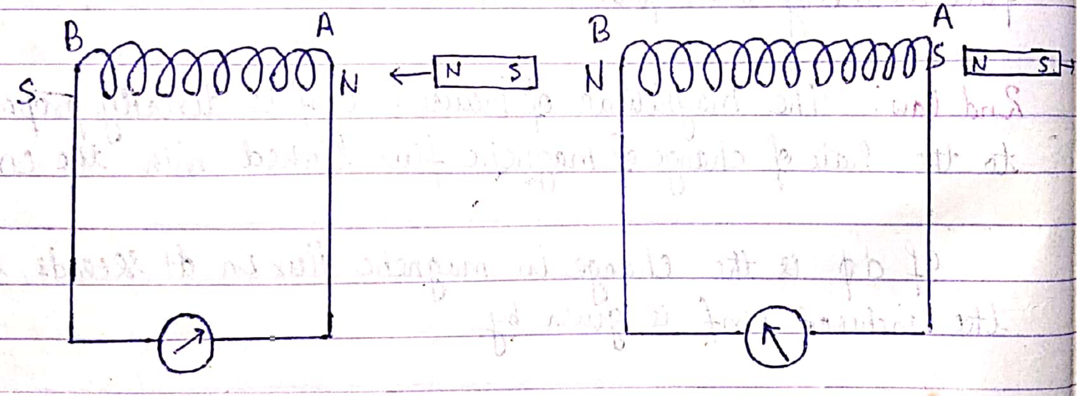
If the coil has N number of turns, the total flux linked with the circuit is given by

$$\mathcal{E} = -N\frac{d\phi}{dt} = -N\frac{d}{dt}\oint_S \vec{B}\cdot\vec{ds}.$$

LENZ'S LAW

This law gives the direction of induced emf. It states that the direction of induced emf is such that the induced emf always opposes the cause of its production.

Let us apply this law to the case of a magnet NS approaching a coil consisting of a few turns of wire in fig a



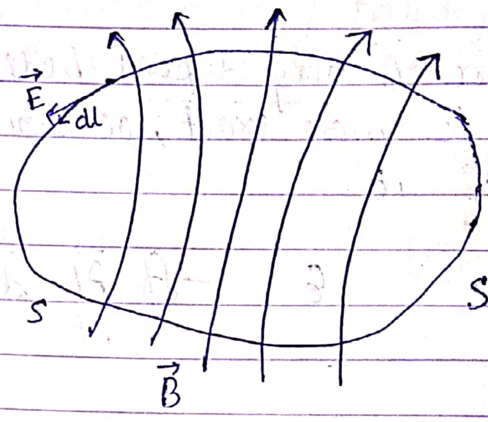
(a) (b)

The induced current set up in the wire as seen from the magnetic side towards the loop. Thus there is a N pole of the coil at A. This repels the N-pole of the magnet, thus opposes the motion of the magnet as predicted by Lenz's law. Similarly when the magnet NS with N pole towards the coil is pulled away, the direction of the induced current is clockwise and there is a S pole of the coil at A and again opposes the pull. So we observe that in all cases, whether the magnet is moved towards or away from the coil, the direction of induced emf is such as to oppose the motion of the magnet, as predicted by Lenz's law.

FARADAY's laws in Integral and Differential form.

INTEGRAL FORM.

Emf is defined as the work done in moving a unit positive charge completely round the circle



i.e $\mathcal{E} = \oint \vec{E}\cdot\vec{dl}$

From Faraday's law

$$\mathcal{E} = -\frac{d\phi}{dt}$$

& $$\oint \vec{E}\cdot\vec{dl} = -\frac{d\phi}{dt}$$

which is Faraday's law in integral form.

DIFFERENTIAL FORM (LAW OF ELECTROMAGNETIC INDUCTION in UNIVERSAL FORM)

Faraday's second law of electromagnetic induction says that the induced emf depends only on the rate of change of flux linked with the circuit. This can be stated as a universal relation.

Consider a closed loop enclosing an area S placed in a magnetic field $\vec{B}$. The total magnetic flux linked with the loop is

$$\phi = \oiint_S \vec{B}\cdot\vec{ds} \quad \text{—— (1)}$$

If $\frac{d\phi}{dt}$ is the rate of change of magnetic flux, then the induced e.m.f is given by

$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}\oint\!\!\oint_S \vec{B}\cdot\vec{ds}$$

or $$\mathcal{E} = -\oint\!\!\oint_S \frac{d\vec{B}}{dt}\cdot\vec{ds}$$

Since $\vec{B}$ may depend both on position and time, hence to be more exact, we should write partial differentiation of $\vec{B}$ i.e.

$$\mathcal{E} = -\oint\!\!\oint \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds} \quad — (2)$$

The magnetic flux changing through a loop induces an electric field around it. And as we know Emf is defined as the work done in moving a unit positive charge completely round a circle.

or $$\mathcal{E} = \oint \vec{E}\cdot\vec{dl} \quad — (3)$$

from (2) and (3)

$$\oint \vec{E}\cdot\vec{dl} = -\oint\!\!\oint \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds} \quad — (4)$$

According to Stokes theorem

$$\oint \vec{E}\cdot\vec{dl} = \oint\!\!\oint (\vec{\nabla}\times\vec{E})\cdot\vec{ds} \quad — (5)$$

from (4) and (5)

$$\oiint_S (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = -\oiint \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

or $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{—— (6)}$$

Eqn (6) represents the differential form of Faraday's law of electromagnetic induction. It is one of the MAXWELL'S equations for electromagnetic field.

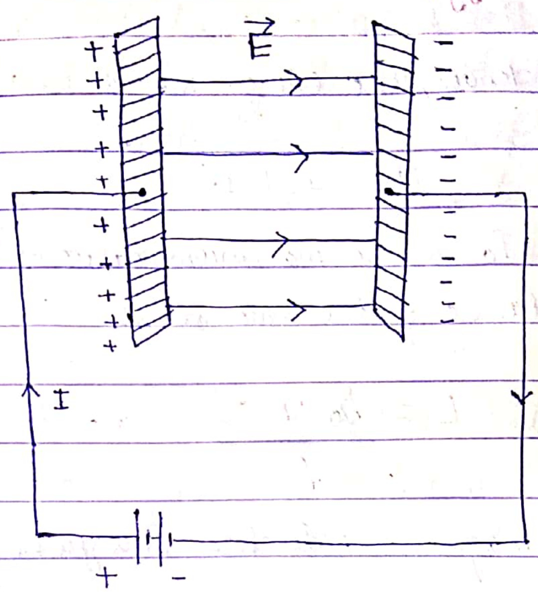
If $\vec{B}$ is constant with time, then

$$\frac{\partial \vec{B}}{\partial t} = 0$$

& $\vec{\nabla} \times \vec{E} = 0$

DISPLACEMENT CURRENT

Consider a parallel plate capacitor of capacitance C connected across a battery. According to Kirchoff's law, the total current entering into any part of the circuit is equal to the total current leaving out of that part of the circuit. But this law is not obeyed in the circuit shown in fig. The current I enters the left plate of the capacitor but no current comes out of this plate. Similarly, current I leaves the right plate of the capacitor but no current enters into this plate.



Maxwell showed that Kirchoff's law can be obeyed by this circuit having a capacitor if some sort of current is associated in the gap of the plates of the capacitor.

When the capacitor is charged, the electric field in the gap between the plates of the capacitor increases. Maxwell gave an idea that a current known as displacement current is associated with the rate of change of electric field between the plates of the capacitor.

Let Q be the charge deposited on the plates of the capacitor at any instant. The electric field between the plates of a parallel plate capacitor is given by

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}, \qquad \left(\because \sigma = \frac{Q}{A}\right)$$

$$\therefore \partial E = \frac{\partial Q}{\epsilon_0 A}$$

But $\partial Q = I\,\partial t$

$\therefore \quad \partial E = \dfrac{I\,\partial t}{\epsilon_0 A}$

or $\quad \dfrac{\partial E}{\partial t} = \dfrac{I}{\epsilon_0 A}$

or $\quad I = \epsilon_0 A \dfrac{\partial E}{\partial t}.$

Which is the displacement current

$\therefore$ Displacement current density, $J_d = \dfrac{I}{A}$

or $\quad J_D = \epsilon_0 \dfrac{\partial E}{\partial t}$

- Displacement current, $I_d = \epsilon_0 A \dfrac{\partial E}{\partial t}$ is due to the change of electric field in the region of the plates of the capacitor. On the other hand, conduction current is due to flow of electrons in the conductor in a particular direction.

- Displacement current serves the purpose to make the total current continuous across the discontinuity in a conduction current

- Displacement current in a good conductor is negligible as compared to the conduction current.

STATIONARY OR STEADY CURRENTS – CONTINUITY EQUATION.

The relation b/w current and current density is given by

$$I = \oint\!\!\oint_S \vec{J}\cdot d\vec{S} = \frac{dq}{dt} \text{ (rate of flow of charge)}$$

If the value of current density $\vec{J}$ remains unchanged with time the current is said to be steady or stationary

Consider a closed surface enclosing volume V. If ρ is the charge density for an infinitesimal volume dv, then $\iiint_V \rho\, dv$ represents the total charge inside the volume V.

Also according to law of conservation of charge, the rate of flow of charge through the closed surface is equal to rate of decrease of charge in it.

i.e. $$\oiint_S \vec{J}\cdot\vec{ds} = -\frac{d}{dt}\iiint_V \rho\, dv = -\iiint_V \left(\frac{\partial\rho}{\partial t}\right) dv \quad \text{—(i)}$$

According to Gauss's divergence theorem.

$$\oiint \vec{J}\cdot\vec{ds} = \iiint_V (\vec{\nabla}\cdot\vec{J})\, dv \quad \text{—— (ii)}$$

From relation (i) and (ii) we have

$$\iiint_V \vec{\nabla}\cdot\vec{J}\, dv = \iiint_V \left(-\frac{\partial\rho}{\partial t}\right) dv$$

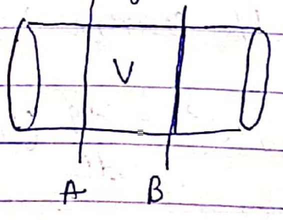
Since the above relation holds good for any arbitrary volume dv. We can write

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \quad \text{—— (iii)}$$

or $$\text{div}\,\vec{J} + \frac{\partial\rho}{\partial t} = 0 \quad \text{—— (iv)}$$

Eq (iv) is called equation of Continuity and represents the physical fact of conservation of charge.

When a steady current flows, the electric charges do not accumulate at any point. This means that the total amount of charge entering volume V through section A is same as the total



amount of charge leaving the volume V through section B. Thus there is no change in the charge density in the volume V

or $\frac{\partial\rho}{\partial t} = 0$

Hence from eqn (iv), $div \vec{J} = 0$ for steady currents
or $\vec{\nabla} \cdot \vec{J} = 0$ —(v).

MODIFICATION OF AMPERE'S CIRCUITAL LAW

According to Ampere's circuital law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \qquad —(1)$$

Since $\oint_C \vec{B} \cdot d\vec{l} = \iint_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{s}$ [Stokes theorem]

and $I = \iint_S \vec{J} \cdot d\vec{s}$

$$\therefore \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad — (2)$$

Taking divergence both sides

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \cdot \mu_0 \vec{J} = \mu_0 (\vec{\nabla} \cdot \vec{J})$$

Since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$

$\therefore \mu_0 (\vec{\nabla} \cdot \vec{J}) = 0$

or $\vec{\nabla} \cdot \vec{J} = 0$ $\quad (\because \mu_0 \neq 0)$

According to continuity eqn $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$.

Since $\vec{\nabla} \cdot \vec{J} = 0$, therefore $\frac{\partial \rho}{\partial t}$ must be zero.

But $\frac{\partial \rho}{\partial t}$ can not be zero for varying current. Thus Maxwell's eqn $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ is valid only for steady current and not for varying current. Hence this eqn needs modification for varying currents.

Maxwell suggested that eqn (2) should be of the form given below:

i.e. $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} +$ something. — (3)

To know this 'something', Maxwell proposed that there is a production of magnetic field due to changing electric field in the same manner as an electric field is produced due to a changing magnetic field.

According to Faraday's law of electromagnetic induction, electric field is produced by a changing magnetic field

ie $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Therefore as suggested by Maxwell, a magnetic field is produced by the changing electric field which is given by

$$|\vec{\nabla} \times \vec{B}| = \mu_0 \epsilon_0 \frac{\partial \phi_E}{\partial t} \quad \text{—— (4)}$$

where $\frac{\partial \phi_E}{\partial t}$ is the rate of change of electric flux. The constant

$\mu_0 \epsilon_0$ has been introduced so that eqn (4) is dimensionally consistent.

Now $\phi_E = EA$

$$\therefore \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} A$$

$$= \mu_0 \epsilon_0 A \frac{\partial \vec{E}}{\partial t} = \mu_0 A \frac{\partial}{\partial t}(\epsilon_0 \vec{E}) \quad \text{—— (5)}$$

Since $\vec{D} = \epsilon_0 \vec{E}$, displacement vector in free space

$$\vec{\nabla} \times \vec{B} = \mu_0 A \frac{\partial \vec{D}}{\partial t} \quad \text{—— (6)}$$

If $A = 1$, then $\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$ — (7)

where $\frac{\partial \vec{D}}{\partial t}$ is called displacement current density using the term as 'something' in eqn (3) we get

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \frac{\partial \vec{D}}{\partial t} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \quad \text{--- (8)}$$

Which is modified form of Ampere's circuital law for varying current and is known as Maxwell's Ampere's circuital law.

Maxwell's Equations or Maxwell's Field Equations:

These are four fundamental equation of electromagnetism and corresponds to a generalisation of certain experimental observations regarding electricity and magnetism. The following four laws of electricity and magnetism constitutes the so called differential form of maxwell's equations:

(i) Gauss' law for the electric field of charge yields

$$\text{div}\,\vec{D} = \vec{\nabla}\cdot\vec{D} = \rho$$

Where $\vec{D}$ is electric displacement in Coulombs/m^2 and ρ is the free charge density in Coul/m^3.

(ii) Gauss's law for magnetic field yields

$$\text{div}\,\vec{B} = \vec{\nabla}\cdot\vec{B} = 0$$

where $\vec{B}$ is magnetic induction in web/m^2.

(iii) Ampere's law in circuital form for the magnetic field accompanying a current when modified by Maxwell yields

$$\text{curl}\,\vec{H} = \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

where $\vec{H}$ is the magnetic field intensity in amperes/m and $\vec{J}$ is current density in amp/m^2.

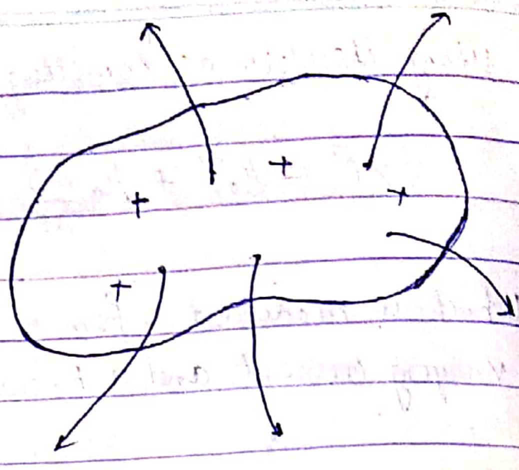
(iv) Faraday's law in circuital form for induced e.m.f produced by rate of change of magnetic flux linked with path yields

$$\text{curl}\,\vec{E} = \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Where $\vec{E}$ is the electric field intensity in Volts/m.

Derivations

First eqn



(i) Let us consider a surface S bounding a volume τ within a dielectric. Originally the volume τ contains no net charge but we allow the dielectric to be polarised say by placing it in an electric field. We also deliberately place some charge on the dielectric body. Thus we have two types of charges:

(a) real charge density ρ (b) bound charge density ρ'

$$\rho' = -\vec{\nabla}\cdot\vec{P}.$$

{ $\vec{P}$ is defined as induced dipole moment per unit volume called electric polarisation vector $\vec{P} = \frac{\vec{p_i}}{V}$, where $\vec{p_i} = q_i\vec{d}$.

Gauss' law can be written as

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\int (\rho+\rho')\, d\tau.$$

i.e. $$\epsilon_0 \oint_S \vec{E}\cdot d\vec{s} = \int_\tau \rho\, d\tau + \int \rho'\, d\tau. \quad -(1)$$

as $\oint \vec{E}\cdot d\vec{s} = \int (\vec{\nabla}\cdot\vec{E})\, d\tau$

and the bound charge density ρ' is defined as $\rho' = -\vec{\nabla}\cdot\vec{P}$

so eqn (1) becomes

$$\epsilon_0 \int_\tau (\vec{\nabla}\cdot\vec{E})\, d\tau = \int \rho\, d\tau - \int_\tau \vec{\nabla}\cdot\vec{P}\, d\tau.$$

i.e. $$\int_\tau \vec{\nabla}\cdot(\epsilon_0\vec{E}+\vec{P})\, d\tau = \int \rho\, d\tau.$$

using $\vec{D} = \epsilon_0\vec{E}+\vec{P}$.

$$\int_\tau (\vec{\nabla}\cdot\vec{D})\, d\tau = \int \rho\, d\tau.$$

or $$\boxed{\vec{\nabla}\cdot\vec{D} = \rho.}$$

Derivation of 2nd eqn.

According to Biot Savart's law

$$\vec{B} = \frac{\mu_0 i}{4\pi} \int \frac{\vec{dl} \times \vec{r}}{r^3}$$

so $$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0 i}{4\pi} \int \vec{\nabla} \cdot \left(\frac{\vec{dl} \times \vec{r}}{r^3} \right) \quad \text{— (1)}$$

using identity

$$\vec{\nabla} \cdot (\vec{A} \times \vec{C}) = \vec{C} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{C})$$

Now $$\vec{\nabla} \cdot \left(\frac{\vec{dl} \times \vec{r}}{r^3} \right) = \frac{\vec{r}}{r^3} \cdot (\vec{\nabla} \times \vec{dl}) - \vec{dl} \cdot \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right)$$

$\left(\text{taking } \vec{A} = \vec{dl} \text{ and } \vec{C} = \frac{\vec{r}}{r^3} \right)$

from eqn (1)

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0 i}{4\pi} \int \left[\frac{\vec{r}}{r^3} (\vec{\nabla} \times \vec{dl}) - \vec{dl} \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right) \right]$$

Now, $\vec{\nabla} \times \vec{dl} = 0$, since $\vec{dl}$ is a constant vector

and $$\frac{\vec{r}}{r^3} = -\vec{\nabla}\left(\frac{1}{r}\right) = -\text{grad}\left(\frac{1}{r}\right)$$

and $$\text{curl } \frac{\vec{r}}{r^3} = -\text{curl grad}\left(\frac{1}{r}\right) = 0$$

so $$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

Derivation of 3rd eqn

According to Ampere's circuital law

$$\oint \vec{H}\cdot d\vec{l} = I \qquad \left[\oint \vec{B}\cdot d\vec{l} = \mu_0 I \text{ and } \vec{B} = \mu_0 \vec{H}\right]$$

or $$\oint_c \vec{H}\cdot d\vec{l} = \int_s \vec{J}\cdot d\vec{s}$$

Changing line integral into surface integral

$$\int_s \text{curl } \vec{H}\cdot d\vec{s} = \int \vec{J}\cdot d\vec{s}$$

or $$\text{Curl } \vec{H} = \vec{J}$$

or $$\vec{\nabla}\times\vec{H} = \vec{J}$$

The difficulty with this eqn is

$$\text{div}(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J}$$

$$\text{div}(\vec{\nabla}\times\vec{H}) = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{J} = 0$$

and from eqn of continuity $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$

Therefore maxwell realised that the above eqn is incomplete for changing electric field and assumed

$$\text{curl } \vec{H} = \vec{J} + \vec{J}_d$$

here $\vec{J}_d$ is called the displacement current density

Now $$\text{div curl } \vec{H} = \text{div } \vec{J} + \text{div } \vec{J}_d$$

$$0 = \text{div } \vec{J} + \text{div } \vec{J}_d$$

or $$\text{div } \vec{J}_d = -\text{div } \vec{J} = \frac{\partial \rho}{\partial t}$$

or $$\text{div } \vec{J}_d = \frac{\partial (\text{div } \vec{D})}{\partial t} \qquad \left[\because \text{div } \vec{D} = \rho \quad \text{1st Maxwell's eqn}\right]$$

or $$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

So $\boxed{\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$

Derivation of 4th eqn.

According to Faraday's law of electromagnetic induction we know that induced e.m.f is proportional to rate of change of flux i.e

$$\mathcal{E} = -\frac{d\phi_B}{dt} \quad \text{—— (1)}$$

Now if $\vec{E}$ be the electric intensity at a point the work done in moving a unit charge through a small distance $\vec{dl}$ is $\vec{E}\cdot\vec{dl}$. So the work done in moving the unit charge once round the circuit $\oint_c \vec{E}\cdot\vec{dl}$. Now as e.m.f is defined as the amount of work done in moving a unit charge once round the electric circuit

$$\mathcal{E} = \oint_c \vec{E}\cdot\vec{dl} \quad \text{— (2)}$$

from (1) and (2)

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\phi_B \quad \text{—— (3)}$$

But as $\phi_B = \int \vec{B}\cdot\vec{ds}$

$$\text{So} \quad \oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int \vec{B}\cdot\vec{ds}$$

Changing line integral into surface integral by Stokes theorem

$$\int_S \text{curl}\,\vec{E}\cdot\vec{ds} = -\frac{d}{dt}\int \vec{B}\cdot\vec{ds}$$

Assuming that surface S is fixed in space and only $\vec{B}$ changes with time, above eqn yields

$$\int_S \left(\text{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right)\cdot\vec{ds} = 0$$

As the above integral is true for any arbitrary surface the integrand must vanish

ie ~~Curl E~~ Curl E $= -\frac{\partial B}{\partial t}$.

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Physical Significance of Maxwell's equations (or Integral form)

By means of Gauss' and Stokes' Theorems we can write the Maxwell's field equations in integral form and hence obtain their physical significance.

(i) Integrating Maxwell's first equation div $D = \rho$ over an arbitrary volume τ we get

$$\int_\tau \vec{\nabla}\cdot\vec{D}\,d\tau = \int_\tau \rho\, d\tau$$

Changing the vol. integral of L.H.S into surface integral by Gauss' divergence theorem and keeping in mind $\int \rho\, d\tau = q$ we get

$$\oint \vec{D}\cdot\vec{ds} = q$$

So Maxwell's first equation signifies that the total flux of electric displacement linked with a closed surface is equal to the total charge enclosed by the closed surface.

(ii) Integrating Maxwell's second equation div $\vec{B} = 0$ over an arbitrary volume τ we get

$$\int_\tau \vec{\nabla}\cdot\vec{B}\, d\tau = 0$$

Converting the vol. integral into surface integral with the help of Gauss' theorem we get

$$\oint \vec{B}\cdot\vec{ds} = 0$$

So Maxwell's 2nd equation signifies that total flux of magnetic induction linked with a closed surface is zero.

(iii) Integrating Maxwell's 3rd equation $\text{curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ over a surface S bounded by the loop C, we get

$$\oint \vec{H} \cdot d\vec{s} = \int \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{s}$$

Converting the surface integral of L.H.S into line integral with the help of Stokes theorem we get

$$\oint H \cdot dl = \int \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{s}$$

Which signifies that magnetomotive force around a closed path $\oint \vec{H} \cdot d\vec{l}$ is equal to the Conduction current plus displacement current linked with that path.

(iv) Integrating Maxwell's 4th equation $\text{curl } \vec{E} = -(\partial \vec{B}/\partial t)$ over a surface S bounded by the loop C we get

$$\int \text{curl } \vec{E} \cdot d\vec{s} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

Converting the surface integral of L.H.S into line integral with the help of Stokes' Theorem we get

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s}$$

which signifies that electromotive force ie line integral of electric intensity around a closed path is equal to the negative rate of change of magnetic flux linked with the path.

Maxwell's equations Particular Cases.

(i) In a Conducting medium of relative permittivity ϵ_r and permeability μ_r as

$$\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E}$$

and $\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$

Maxwell's equation reduces to

(I) $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_r \epsilon_0$ (II) $\vec{\nabla} \cdot \vec{H} = 0$

(III) $\vec{\nabla} \times \vec{H} = \vec{J} + \epsilon_r \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ (IV) $\vec{\nabla} \times \vec{E} = -\mu_r \mu_0 \dfrac{\partial \vec{H}}{\partial t}$

(ii) In a ~~co~~ non Conducting media of relative permittivity ϵ_r and permeability μ_r as

$$\rho = \sigma = 0$$

so $\vec{J} = \sigma \vec{E} = 0$

and hence Maxwell's eqns become

(I) $\vec{\nabla} \cdot \vec{E} = 0$ (II) $\vec{\nabla} \cdot \vec{H} = 0$

(III) $\vec{\nabla} \times \vec{H} = \epsilon_r \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ (IV) $\vec{\nabla} \times \vec{E} = -\mu_r \mu_0 \dfrac{\partial \vec{H}}{\partial t}$

(iii) In free space as

$$\epsilon_r = \mu_r = 1,$$
$$\rho = \sigma = 0, \quad \vec{J} = \sigma \vec{E} = 0.$$

Maxwell's equations become

(I) $\vec{\nabla} \cdot \vec{E} = 0$ (II) $\vec{\nabla} \cdot \vec{H} = 0$

(III) $\vec{\nabla} \times \vec{H} = \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ (IV) $\vec{\nabla} \times \vec{E} = -\mu_0 \dfrac{\partial \vec{H}}{\partial t}$

Boundary Conditions

The electric and magnetic fields at any surface of discontinuity obey following boundary condition.

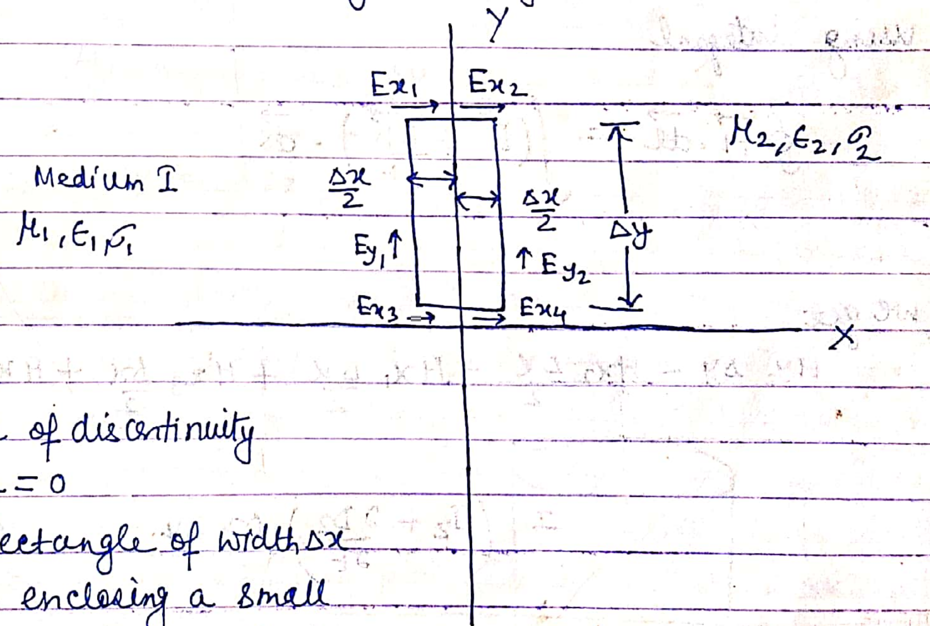
(i) The tangential component of $\vec{E}$ is continuous at the surface ie it is same just outside or inside the surface.

(ii) The tangential component of $\vec{H}$ is continuous across a surface except at the surface of perfect conductor. At the surface of perfect conductor the tangential component of $\vec{H}$ is discontinuous by amount equal to the surface current per unit width.

(iii) The normal component of $\vec{B}$ is continuous at the surface of discontinuity.

(iv) The normal component of $\vec{D}$ is continuous if there is no surface charge density. Otherwise $\vec{D}$ is discontinuous by an amount equal to surface charge density.

Proof.



Let the surface of discontinuity be the plane $x = 0$

A small rectangle of width Δx and length Δy enclosing a small area of each media I and II is considered as shown in fig.

Using integral form of Maxwell's eqns.

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

For elementary rectangle

$$E_{y_2} \cdot \Delta y - E_{x_2} \cdot \frac{\Delta x}{2} - E_{x_1} \frac{\Delta x}{2} - E_{y_1} \Delta y + E_{x_3} \frac{\Delta x}{2}$$

$$+ E_{x_4} \frac{\Delta x}{2} = -\frac{\partial B_z}{\partial t} \Delta x \Delta y$$

where B_z is the average magnetic flux density through rectangle.

Let $\Delta x \to 0$ & surface of discontinuity lies betwn sides of rectangle. Here B and E are always finite and

$$E_{y_2} \Delta y - E_{y_1} \Delta y = 0$$

or $\quad E_{y_1} = E_{y_2}$

ie tangential component of $\vec{E}$ is continuous across the boundary.

using integral

$$\oint \vec{H} \cdot \vec{dl} = \int \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{ds}$$

we get

$$H_{y_2} \Delta y - H_{x_2} \frac{\Delta x}{2} - H_{x_1} \frac{\Delta x}{2} + H_{x_3} \frac{\Delta x}{2} + H_{x_4} \frac{\Delta x}{2}$$

$$= \left(J_z + \frac{\partial D_z}{\partial t} \right) \Delta x \Delta y \quad — (1)$$

For $\Delta x \to 0$

$$H_{y_2} \Delta y - H_{y_1} \Delta y = 0$$

or $\quad H_{y_1} = H_{y_2}$

* $J = \frac{E}{\rho}$). $J = \sigma E$, $\sigma = \frac{1}{\rho}$ where $\rho = \frac{1}{\sigma}$

$E = \frac{J}{\sigma}$ σ is the conductivity of the substance.

i.e Tangential component of H is continuous for finite current density.

Case of perfect Conductor.

A perfect Conductor is one which has infinite conductivity. In such a Conductor the electric intensity $\vec{E}$ is zero for any finite current density. In a good conductor high frequency current will flow near the surface. This gives rise to current sheet in which finite current per unit width J_s amp/m flows in the sheet of very small depth Δx but with the required infinitely large current density J.

If J_z becomes infinite, right hand side of the eqn will not become zero.

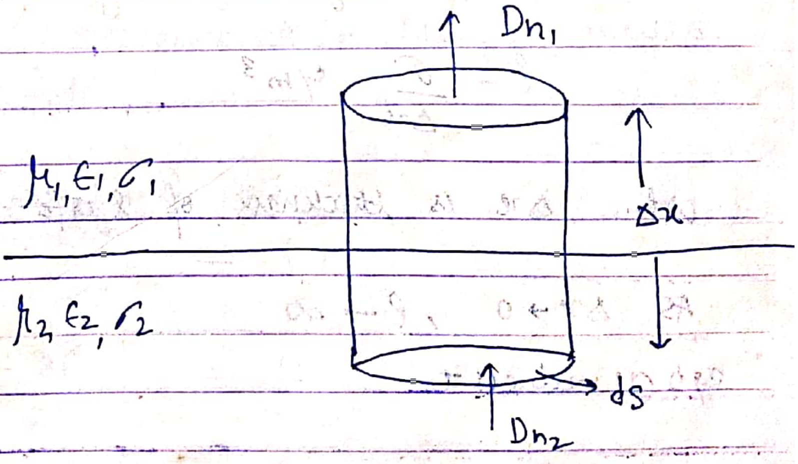
let J_s amp/m will be actual current per unit width flowing in the surface.

For $\Delta x \to 0$ eqn (1) becomes

$$H_{y_2} \cdot \Delta y - H_{y_1} \Delta y = J_{sz} \Delta y$$

$$H_{y_2} - H_{y_1} = J_{sz}$$

Conditions for normal component of $\vec{B}$ and $\vec{D}$



using eqn

$$\vec{\nabla} \cdot \vec{D} = \rho$$

and its integral form

$$\oint \vec{D} \cdot \vec{ds} = \int_V \rho \, dv$$

Applying to the pill box as shown in fig.

$$D_{n_1} ds - D_{n_2} ds + \psi_{edge} = \rho \Delta x \, ds \quad -②$$

where ds is area of each of the flat surface of pill box having separation Δx & ρ is average charge density within vol$^{\underline{m}}$ $\Delta x \, ds$

here, ψ_{edge} is outward electric flux through the curved edge of the pill box

as $\Delta x \to 0$, $\psi_{edge} = 0$

Hence for finite value of ρ as $\Delta x \to 0$

$$D_{n_1} ds - D_{n_2} ds = 0$$

or $\quad D_{n_1} = D_{n_2}$

i.e the normal component of $\vec{D}$ is continuous across the surface provided there is no surface charge.

In case of metallic surface the charge is considered to reside on the surface. If this layer has ρ surface charge density σ Coulomb/m^2 the charge density ρ on the surface layer is given by

$$\rho = \frac{\sigma}{\Delta x} \quad C/m^3.$$

where Δx is thickness of surface layer

As $\Delta x \to 0$, $\rho \to \infty$

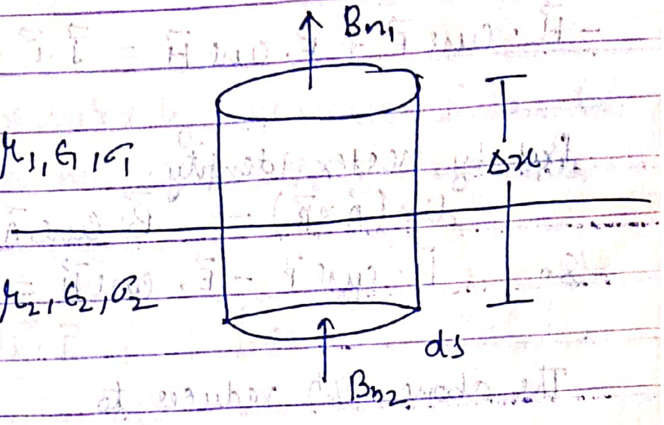
eq$^{\underline{n}}$ (2) reduces to

$$D_{n_1} ds - D_{n_2} ds = \frac{\sigma}{\Delta x} \Delta x \, ds$$

or $D_{n_1} - D_{n_2} = \sigma$.

using integral form of eqn
$\vec{\nabla}\cdot\vec{B} = 0$

ie $\oint_S \vec{B}\cdot d\vec{s} = 0$



Applying to pill box.

$$B_{n_1} ds - B_{n_2} ds + \psi_{edge} = 0$$

as $\Delta x \to 0$ $\psi_{edge} = 0$

& so $B_{n_1} = B_{n_2}$

ie normal component of magnetic flux density is always continuous across a boundary surface.

Energy in Electromagnetic fields (Poynting's theorem)

From maxwells' equations it is possible to derive an important expression which we shall recognise as the energy principle in an electromagnetic field.

For this we consider Maxwells eqns

$$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{--- (1)}$$

and

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (2)}$$

If we take the scalar product of eqn (1) with $\vec{E}$ and of eqn (2) with $(-\vec{H})$ we get

$$\vec{E}\cdot \text{curl}\,\vec{H} = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} \quad \text{--- (3)}$$

and $\quad -\vec{H}\cdot \text{curl}\,\vec{E} = +\vec{H}\cdot\dfrac{\partial \vec{B}}{\partial t}$ — (4)

adding eqns (3) and (4) we get

$$-\vec{H}\cdot\text{curl}\,\vec{E} + \vec{E}\cdot\text{curl}\,\vec{H} = \vec{J}\cdot\vec{E} + \left[\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right]$$

But by vector identity.

$$\text{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\text{curl}\,\vec{A} - \vec{A}\cdot\text{curl}\,\vec{B}$$

So $\quad \vec{H}\cdot\text{curl}\,\vec{E} - \vec{E}\cdot\text{curl}\,\vec{H} = \text{div}(\vec{E}\times\vec{H})$

∴ The above eqn reduces to

$$-\text{div}(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \left[\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right] \quad -(5)$$

Now $\quad \vec{E}\cdot\dfrac{\partial \vec{D}}{\partial t} = \epsilon_r\epsilon_0\vec{E}\dfrac{\partial \vec{E}}{\partial t} = \dfrac{1}{2}\epsilon_r\epsilon_0\dfrac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) = \dfrac{1}{2}\dfrac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$

and $\quad \vec{H}\cdot\dfrac{\partial \vec{B}}{\partial t} = \mu_r\mu_0\vec{H}\dfrac{\partial \vec{H}}{\partial t} = \dfrac{1}{2}\mu_r\mu_0\dfrac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) = \dfrac{1}{2}\dfrac{\partial}{\partial t}(\vec{H}\cdot\vec{B})$

So eqn (5) reduces to

$$\vec{J}\cdot\vec{E} + \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}) + \text{div}(\vec{E}\times\vec{H}) = 0 \quad -(6)$$

Each term in above eqn. can be given some physical meaning if it is multiplied by an element of volume $d\tau$ and integrated over a volume τ whose enclosing surface is S. Thus the result is

$$\int_\tau (\vec{J}\cdot\vec{E})\,d\tau + \int_\tau \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{B} + \vec{H}\cdot\vec{B})\,d\tau + \int_\tau \text{div}(\vec{E}\times\vec{H})\,d\tau = 0$$

But as $\displaystyle\int_\tau \text{div}(\vec{E}\times\vec{H})\,d\tau = \oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}$

So $\displaystyle\int_\tau (\vec{J}\cdot\vec{E})\,d\tau + \int_\tau \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{B} + \vec{H}\cdot\vec{B})\,d\tau + \oint_S (\vec{E}\times\vec{H})\cdot d\vec{S} = 0$

— (A)

To understand what eqn (A) means, let us now interpret various term in it.

(A) Interpretation of $\int_\tau \vec{J}\cdot\vec{E}\, d\tau$:

The current distribution represented by the vector $\vec{J}$ can be considered as made up of various charges q_i moving with velocity V_i so that

$$\int \vec{J}\cdot\vec{E}\, d\tau = \int I\, d\vec{l}\cdot\vec{E} \qquad [\text{as } \vec{J}\, d\tau = I\, d\vec{l}]$$

$$= \int dq\, \vec{v}\cdot\vec{E} \qquad [\text{as } I dl = (dq/dt)\, dl = dq\, v$$

$$= \sum q_i (\vec{V_i}\cdot\vec{E_i}) \qquad \text{— (7)}$$

where $\vec{E_i}$ denotes the electric field at the position of charge q_i.

Now electromagnetic force on the ith charged particle is given by the Lorentz expression

$$\vec{F_i} = q_i(\vec{E_i} + \vec{V_i}\times\vec{B_i})$$

So the work done per unit time on the charge q_i by the field will be

$$\frac{\partial W_i}{\partial t} = \vec{F_i}\cdot\vec{V_i} \qquad \left[\because \frac{dW}{dt} = \frac{\vec{F}\cdot d\vec{l}}{dt} = \vec{F}\cdot\vec{V}\right]$$

$$= q_i(\vec{E_i} + \vec{V_i}\times B_i)\cdot\vec{V_i} \qquad (\text{as } \vec{F_i} = q_i(\vec{E_i} + \vec{V_i}\times\vec{B_i}))$$

ie $$\frac{\partial W_i}{\partial t} = q_i \vec{V_i}\cdot\vec{E_i} \qquad [\text{as } \vec{V_i}\cdot(\vec{V_i}\times\vec{B_i}) = (\vec{V_i}\times\vec{V_i})\cdot\vec{B_i} = 0$$

So the rate at which the work is done by the field on the charge is $\frac{\partial W}{\partial t} = \sum \frac{\partial W_i}{\partial t} = \sum q_i \vec{V_i}\cdot\vec{E_i}$ — (8)

Comparing eqn (7) and (8) we find that

$$\int \vec{J}\cdot\vec{E}\, d\tau = \frac{dW}{dt} \quad -(9)$$

ie the first term $\int (\vec{J}\cdot\vec{E})\, d\tau$ represents the rate at which work is done by the field on the charges.

(B) Interpretation of $\int_\tau \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\, d\tau$

If we allow the volume τ to be arbitrary large the surface integral in eqn (A) can be made to vanish by placing the surface S sufficiently far away so that the field can not propagate to this distance in any finite time ie $\oint_S \vec{E}\times\vec{H}\cdot d\vec{s} = 0$
So under these circumstances eqn (A) reduces to

$$\frac{\partial}{\partial t}\int_{\text{all space}} \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}) + \frac{\partial W}{\partial t} = 0$$

ie $$\frac{\partial}{\partial t}\left[\int_{\text{all space}} \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\, d\tau + W\right] = 0$$

Thus the quantity in the square bracket is conserved. Now consider a closed system in which the total energy is assumed to be constant. The system consists of the electromagnetic field and all the charged particles present in the field. The term W represents the total kinetic energy of the particles. We are therefore led to associate the remaining energy term

$$\int_{\text{all space}} \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\, d\tau$$

with the energy of electro-magnetic field ie

$$U = \int_{\text{all space}} \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\, d\tau \quad -(10)$$

the quantity U may be considered to be a kind of potential energy. One need to ascribe this potential energy to the charged particles and must consider this term as a field energy. A concept such as energy stored in the field itself rather than residing with the particles is a 'basic concept of theory of electromagnetism.

Note:-

If we write eqn (10) as

$$U = \int_{all\,space} u\, d\tau$$

where $u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})$ may be thought of as the energy density of the electromagnetic field.

Further as

$$u = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}$$

first term on R.H.S contains only electrical quantities while the second one magnetic, we can have

$$u = u_e + u_m$$

where $u_e = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{2}\epsilon_r\epsilon_0 E^2$ = energy density of electric field

and $u_m = \frac{1}{2}\vec{H}\cdot\vec{B} = \frac{1}{2}\mu_r\mu_0 H^2$ = energy density of magnetic field.

(C) Interpretation of $\oint_s \vec{E}\times\vec{H}\cdot d\vec{s}$

Instead of taking the volume integral in eqns (A) over all space, let us now consider a finite volume. In this case the surface integral of $(\vec{E}\times\vec{H})$ will not in general vanish and so this term must be retained. Let us construct the surface S in such a way that in interval of time under consideration, none of the charged particles will cross this surface. Then for the conservation of energy

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} = -\oint_s (\vec{E}\times\vec{H})\cdot d\vec{s} \quad \text{---(11)}$$

The left hand side is the time rate of change of energy of the field and of the particles contained within the volume τ.

Thus the surface integral $\oint_s (\vec{E}\times\vec{H}).d\vec{s}$ must be considered as the energy flowing out of the volume bounded by the surface S per sec. But by hypotheses no particles are crossing the surface, so the vector $(\vec{E}\times\vec{H})$ is to be interpreted as the amount of the field energy passing through unit area of the surface in unit time which is normal to the direction of energy flow. The vector $(\vec{E}\times\vec{H})$ is called the poynting vector and is represented by S i.e.

$$\vec{S} = \vec{E}\times\vec{H}. \quad -(12)$$

Interpretation of Energy Equation:

In the light of the above eqn (6) in differential form can be written as

$$\vec{J}\cdot\vec{E} + \frac{\partial U}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0 \quad —— (13)$$

In the event that the medium has zero conductivity i.e. $\vec{J} = \sigma\vec{E} = 0$, the above eqn becomes exactly of the same form as the continuity eqn which expresses the law of conservation of charge. We are led by this analogy that the physical meaning of eqs 13, 11 & (A) is to represent the law of conservation of energy for electromagnetic phenomena. According to eqn (11) time rate of change of electromagnetic energy within a certain volume plus the rate at which the work done is done by the field on the charges is equal to the energy flowing into the system through its bounding surface per unit time.

Electromagnetic waves in free space.

We Know that Maxwell's equations are

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}\right\} \quad — (1)$$

Where $\vec{D}$ = electric displacement in Coulomb/m^3

$\vec{B}$ = magnetic induction in Web/m^2

$\vec{H}$ = magnetic field intensity in amperes/m and

$\vec{J}$ = current density in amp/m^2

$\vec{E}$ = electric field intensity in Volts/m.

with $\vec{J} = \sigma\vec{E}$

$\vec{B} = \mu\vec{H}$

$\vec{D} = \epsilon\vec{E}$

and in free space i.e. Vacuum

$\rho = 0, \quad \epsilon_r = 1$

$\sigma = 0 \quad \mu_r = 1$

So the Maxwell's equations reduce to

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0 &&—(a) \\ \vec{\nabla}\cdot\vec{H} &= 0 &&—(b) \\ \vec{\nabla}\times\vec{H} &= \epsilon_0\frac{\partial \vec{E}}{\partial t} &&—(c) \\ \vec{\nabla}\times\vec{E} &= -\mu_0\frac{\partial \vec{H}}{\partial t} &&—(d) \end{aligned}\right\} \quad — (2)$$

Now if

(I) We take the curl of equation 2(c) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \epsilon_0\, \vec{\nabla}\times\left(\frac{\partial \vec{E}}{\partial t}\right)$$

or $$\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H}\right] = \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \quad — (3)$$

But from eqn (2b) and (2d)

$$\vec{\nabla}\cdot\vec{H} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

So equation (3) reduces to

$$\nabla^2\vec{H} - \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \quad \text{with } \mu_0\epsilon_0 = \frac{1}{c^2} \quad — (A)$$

(II) We take the curl of equation 2(d) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\mu_0 \frac{\partial \vec{H}}{\partial t}\right)$$

i.e. $$\left[\nabla(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}\right] = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}) \quad — (4)$$

But from equation 2(a) and 2(c)

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

So eqn (4) reduces to

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \quad \text{with } \mu_0\epsilon_0 = \frac{1}{c^2} \quad — (B)$$

A glance at differential equations (A) and (B) reveals that these are identical in from to the equation

$$\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0 \quad — (5)$$

However equation (5) is a standard wave equation representing unattenuated wave travelling at a speed V. So we conclude that field vector $\vec{E}$ and $\vec{H}$ are propagated in free space as waves at a speed

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \sqrt{\frac{4\pi}{4\pi\epsilon_0 \mu_0}} = \sqrt{9\times10^9} \times \sqrt{10^7}$$

$= 3\times10^8$ m/s. ie the velocity of light.

as $\frac{1}{4\pi\epsilon_0} = 9\times10^9\ Nm^2C^{-2}$.

and $\frac{\mu_0}{4\pi} = 10^{-7}\ TmA^{-1}$

Equations (A) and (B) are vector wave equations their solution can be obtained in many forms, for instance either stationary or progressive waves or having wavefronts of particular types such as plane, cylindrical or spherical. where no boundary conditions are imposed, plane progressive solutions are most appropriate. So as the plane progressive solution of eqn (5) is

$$\psi = \psi_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

∴ the solutions of equations (A) and (B) will be of the form

$$\left.\begin{aligned} \vec{E} &= E_0\, e^{-i(\omega t - \vec{k}\cdot\vec{r})} \\ \vec{H} &= H_0\, e^{-i(\omega t - \vec{k}\cdot\vec{r})} \end{aligned}\right\} \quad -(C)$$

where $\vec{K}$ is the so called wave vector given by

$$\vec{K} = k\vec{n} = \frac{2\pi}{\lambda}\vec{n} = \frac{2\pi f}{c}\vec{n} = \frac{\omega}{c}\vec{n}$$

where $\vec{n}$ is the unit vector in the direction of wave propagation.

The form of field vectors $\vec{E}$ and $\vec{H}$ given by eqn (e) suggests that in case of field vectors operator $\vec{\nabla}$ is equivalent to ~~as~~ $i\vec{k}$ while $\partial/\partial t$ to $(-i\omega)$.

as

$$\psi = \psi_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

$$\psi = \psi_0 e^{-i\omega t} \cdot e^{i(k_x\hat{i} + k_y\hat{j} + k_z\hat{k})\cdot(\hat{i}x + \hat{j}y + \hat{k}z)}$$

$$= \psi_0 e^{-i\omega t} \cdot e^{i(xk_x + yk_y + zk_z)}$$

Now $$\vec{\nabla}\psi = \left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) e^{i(xk_x + yk_y + zk_z)} \cdot e^{-i\omega t}\right\}$$

$$= \left[\hat{i} e^{i(xk_x + yk_y + zk_z)} \times ik_x + \hat{j} e^{i(xk_x + yk_y + zk_z)} \times ik_y + \hat{k} e^{i(xk_x + yk_y + zk_z)} \times ik_z\right] \psi_0 e^{-i\omega t}$$

$$= i\psi_0 e^{-i\omega t}(\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\, e^{i(xk_x + yk_y + zk_z)}$$

& $$\vec{\nabla}\psi = \psi_0 i\vec{k} e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

$$= i\vec{k}\psi$$

so $$\vec{\nabla} \rightarrow i\vec{k}.$$

also $$\psi = \psi_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

so $$\frac{\partial\psi}{\partial t} = -i\omega\,\psi_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

$$= -i\omega\psi$$

so $$\frac{\partial}{\partial t} = -i\omega.$$

ie the operator $\vec{\nabla}$ is equivalent to $i\vec{k}$ while $\frac{\partial}{\partial t}$ to $(-i\omega)$

So various operations are equivalent to

curl $\rightarrow i\vec{k}\times$ div $\rightarrow i\vec{k}\cdot$

grad $\rightarrow i\vec{k}$ $\frac{\partial}{\partial t} = -i\omega$

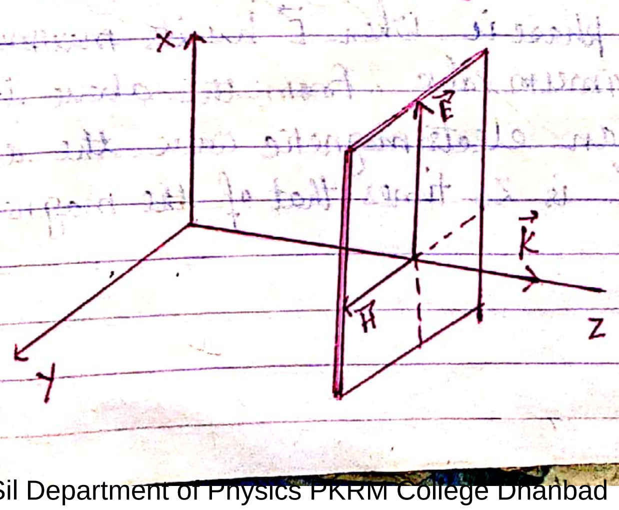
So Maxwell's equations in free space ie eqn (2) in terms of operator $(i\vec{k})$ and $(-i\omega)$ can be written as

$$\left.\begin{aligned} \vec{K}\cdot\vec{E} &= 0 \quad &&-(a) \\ \vec{K}\cdot\vec{H} &= 0 &&-(b) \\ -\vec{K}\times\vec{H} &= \omega\epsilon_0\vec{E} &&-(c) \\ \vec{k}\times\vec{E} &= \omega\mu_0\vec{H} &&-(d) \end{aligned}\right\} \quad -(4).$$

Regarding plane electromagnetic waves in free space it is worthy to note that:

(i) As according to eqn 4(a) the vector $\vec{E}$ is perpendicular to the direction of propagation while according to eqn 4(b) vector $\vec{H}$ is perpendicular to the direction of propagation (ie in an electromagnetic wave both the vectors $\vec{E}$ and $\vec{H}$ are perpendicular to the direction of wave propagation), electromagnetic waves are transverse in nature.

Further as according to eqn 4(d) $\vec{H}$ is perpendicular to both $\vec{E}$ and $\vec{K}$ while according to eqn 4(a) $\vec{E}$ is perpendicular to $\vec{K}$. This all in turn implies that in a plane electromagnetic waves vectors $\vec{E}$, $\vec{H}$ and $\vec{K}$ are orthogonal.



(ii) As according to equation 4(d)

$$\vec{K}\times\vec{E} = \omega\mu_0\vec{H}$$

ie $$\vec{H} = \frac{K}{\omega\mu_0}(\hat{n}\times\vec{E}) \quad (\text{as } \vec{K} = \vec{n}k)$$

ie $$\vec{H} = \frac{\vec{n}\times\vec{E}}{c\mu_0} = c\epsilon_0(\vec{n}\times\vec{E})$$ $$\left[\text{as } k = \frac{\omega}{c} \text{ and } \epsilon_0\mu_0 = \frac{1}{c^2} \quad \text{so } \vec{H} = \frac{\vec{n}\times\vec{E}}{c\mu_0} = \frac{c(\vec{n}\times\vec{E})}{\frac{1}{\epsilon_0\mu_0}\times\mu_0} = c\epsilon_0(\vec{n}\times\vec{E})\right]$$

ie $$\vec{B} = \frac{\vec{n}\times\vec{E}}{c}$$

and as $$\vec{H} = \frac{\vec{n}\times\vec{E}}{c\mu_0}$$

Now this equation in term of module

$$|\vec{H}| = \frac{1}{\mu_0 c}|\vec{E}|$$

so $$\left|\frac{\vec{E}}{\vec{H}}\right| = \frac{E_0}{H_0} = c\mu_0 = \frac{1}{\sqrt{\epsilon_0\mu_0}}\mu_0 = \sqrt{\frac{\mu_0^2}{\epsilon_0\mu_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0$$

$$\left(\text{as } \mu_0\epsilon_0 = \frac{1}{c^2}\right)$$

As the ratio $|\vec{E}/\vec{H}|$ is real and positive, the vectors $\vec{E}$ and $\vec{H}$ are in phase ie when $\vec{E}$ has its maximum value $\vec{H}$ has also its maximum value. From the above it is also clear that in an electromagnetic wave the amplitude of electric vector $\vec{E}$ is Z_0 times that of the magnetic vector $\vec{H}$.

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \left[\sqrt{\frac{4\pi\times10^{-7}}{(1/4\pi\times9\times10^9)}}\right] = 120\pi \simeq 377\,\Omega$$

where the units of Z_0 are most easily seen from the fact that it measures a ratio of E in volt/m to H in amp-turn/m and therefore must equal volt/amp or ohms. ie of impedence, hence it is called the intrinsic or characteristic impedence of free space. This is a constant having value $377\,\Omega$.

(iii) The Poynting vector for a plane electromagnetic wave in free space will be given by

$$\vec{S} = \vec{E}\times\vec{H} = \frac{\vec{E}\times(\vec{n}\times\vec{E})}{c\mu_0}$$

$$\text{ie } \vec{S} = \left[(\vec{E}\cdot\vec{E})\vec{n} - (\vec{E}\cdot\vec{n})\vec{E}\right]\frac{1}{c\mu_0} \quad \left[\because \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\right]$$

$$= \frac{1}{c\mu_0}E^2\vec{n} \quad (\text{as } \vec{E}\cdot\vec{n} = 0 \text{ because } \vec{E} \text{ is } \perp \text{ to } \vec{n})$$

$$\text{or} \quad \vec{S} = \epsilon_0 c E^2\vec{n} = \frac{1}{Z_0}E^2\vec{n} \quad \left(\text{as } \frac{1}{c\mu_0} = c\epsilon_0 = \frac{1}{Z_0}\right)$$

$$\text{or} \quad \langle\vec{S}\rangle = \epsilon_0 c\langle E^2\rangle\vec{n}$$

But as

$$\langle E^2\rangle = \left\langle\left[E_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}\right]_{real}^2\right\rangle = E_0^2\langle\cos^2(\omega t - \vec{k}\cdot\vec{r})\rangle$$

$$\left[\because \langle\cos^2(\omega t - \vec{k}\cdot\vec{r})\rangle = \frac{1}{2} \text{ as } \langle\cos^2\theta\rangle = \frac{1}{2}\right]$$

$$\text{ie } \langle E^2\rangle = \frac{E_0^2}{2} = \frac{E_0}{\sqrt{2}}\frac{E_0}{\sqrt{2}}$$

$$= E_{rms}^2$$

So $\langle \vec{S} \rangle = \epsilon_0 c E^2_{rms} \vec{n} = \frac{1}{Z_0} E^2_{rms} \vec{n}$ --- (E)

ie the flow of energy in a plane wave in free space is in the direction of wave propagation.

(iv) In case of plane electromagnetic wave

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\epsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\epsilon_0}{\mu_0}\left(\frac{E}{H}\right)^2 = 1 \quad \left(\text{as } \frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}}\right.$$

ie the electro~~magnetic~~static energy density is equal to the magnetostatic energy density.

Further

$$\frac{\langle \vec{S} \rangle}{\langle u \rangle} = \frac{\epsilon_0 c E^2_{rms} \vec{n}}{\epsilon_0 E^2_{rms}} = c\vec{n}$$

or $\vec{S} = cu\vec{n}$

as Total electromagnetic energy density

$$u = u_e + u_m = 2u_e = 2 \times \frac{1}{2}\epsilon_0 E^2 = \epsilon_0 E^2$$

$\therefore$ Time average of energy density

$$\langle u \rangle = \langle \epsilon_0 E^2 \rangle = \epsilon_0 \langle (E_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})})^2 \rangle_{real}$$

$$= \epsilon_0 E_0^2 \langle \cos^2(\omega t - \vec{k}\cdot\vec{r}) \rangle = \frac{1}{2}\epsilon_0 E_0^2 = \epsilon_0 E^2_{rms}$$

$\therefore \vec{S} = cu\vec{n}$

This implies that electromagnetic energy in free space is transmitted energy with the speed of light c with which the field vectors $\vec{E}$ and $\vec{H}$ do.

Propagation of E.M.W in Isotropic Dielectrics.

A non-Conducting medium whose properties are same in all directions is called isotropic dielectric.

We know that Maxwell's field equations are

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho \\ \vec{\nabla}\cdot B &= 0 \\ \vec{\nabla}\times\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}\right\} \text{with} \left\{\begin{aligned} \vec{J} &= \sigma\vec{E} \\ \vec{B} &= \mu\vec{H} \\ \vec{D} &= \epsilon\vec{E} \end{aligned}\right. \quad \text{— (1)}$$

and in isotropic dielectrics $\sigma = 0$ and $\rho = 0$.

So Maxwell's equations reduce to

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0 \quad &-(a)\\ \vec{\nabla}\cdot\vec{H} &= 0 \quad &-(b)\\ \vec{\nabla}\times\vec{H} &= \epsilon\frac{\partial\vec{E}}{\partial t} \quad &-(c)\\ \vec{\nabla}\times\vec{E} &= -\mu\frac{\partial\vec{H}}{\partial t} \quad &-(d)\end{aligned}\right\} \quad (2)$$

(I) We take the curl of eqn 2(c) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \epsilon\,\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H} = \epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \quad -(3)$$

But from eqn (2b) and 2(d)

$\vec{\nabla}\cdot\vec{H} = 0$ and $\vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$

So eqn (3) reduces to

$$\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

ie $$\nabla^2\vec{H} - \frac{1}{v^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \text{ with } \mu\epsilon = \frac{1}{v^2} \quad -(A)$$

(II) We take the curl of eqn 2(d) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

ie $$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

But from eqn 2(a) and 2(c)

$$\vec{\nabla}\cdot\vec{E} = 0 \text{ and } \vec{\nabla}\times\vec{H} = \epsilon\frac{\partial\vec{E}}{\partial t}$$

So eq$\underline{n}$ (4) reduces to

$$\nabla^2\vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2\vec{E} - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \text{ with } \mu\epsilon = 1/v^2 \quad \text{—(B)}$$

A glance at eq$\underline{n}$ (A) and (B) reveals that these are identical in form to the eq$\underline{n}$

$$\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \quad \text{— (5)}$$

However eq$\underline{n}$ (5) is a standard wave eq$\underline{n}$ representing an unattenuated wave travelling at a speed v. So we conclude that field vectors $\vec{E}$ and $\vec{H}$ propagate in isotropic dielectric as waves given by

$$\begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = \begin{Bmatrix} \vec{E_0} \\ \vec{H_0} \end{Bmatrix} e^{-i(\omega t - \vec{k}\cdot\vec{r})} \quad \text{—(C)}$$

at a speed

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\epsilon_r \mu_r \epsilon_0 \mu_0}} \quad (\text{as } \epsilon = \epsilon_r\epsilon_0 \text{ and } \mu = \mu_r\mu_0)$$

ie $$v = \frac{c}{\sqrt{\epsilon_r \mu_r}} < c \quad \left[\text{as } \epsilon_0\mu_0 = \frac{1}{c^2};\ \epsilon_r \text{ and } \mu_r > 1\right] \text{—(6)}$$

ie the speed of electromagnetic wave in isotropic dielectrics is less than the speed of electromagnetic waves in free space

Further as index of refraction is defined as

$$n = \frac{c}{v}$$

So in this particular case $n = \sqrt{\epsilon_r \mu_r}$ $\left[\text{as } v = \frac{c}{\sqrt{\mu_r \epsilon_r}}\right.$

and as in non-magnetic medium $\mu_r = 1$

$$n = \sqrt{\epsilon_r} \quad \text{ie } n^2 = \epsilon_r \quad — (7).$$

Equation (7) is called Maxwell's relation and has been actually confirmed by experiments for long waves ie radio frequency and slow infrared oscillations. In visible region of the spectrum this relation is also fairly well satisfied for some substances such as H_2, CO_2, N_2 and O_2. But for many other substances it fails, when as a rule the substance shows infrared selective absorption. With water the failure is especially marked. For water $\mu_r \simeq 1$, $\epsilon_r \simeq 81$ so that $n \simeq 9$. But it is ~~also~~ well known that the index of refraction of water for light is very closely given by 4/3 ie 1.33. The solution of this apparent contradiction lies in the fact that our macroscopic formulation of electromagnetic theory gives no indication of the values to be expected for ϵ_r and μ_r and we must rely on experiment to obtain them. It turns out that these quantities are not really constant for a given material but usually have a strong dependence on frequency due to dispersion.

It is also worthy to note that $\epsilon_r > 1$, the velocity of light in an isotropic dielectric medium

$$v = \frac{c}{n} = \frac{c}{\sqrt{\epsilon_r}}.$$

is always less than c as $\epsilon_r > 1$.

It is therefore possible for high energy particle to have velocities in excess of v. When such particles pass through a dielectric a bluish light known as Cerenkov-radiation is emitted due to interaction of uniformly moving charged particles with the medium.

The form of field vectors $\vec{E}$ and $\vec{H}$ given by equation (c) suggests that

$\vec{\nabla} \to i\vec{K}$ and $\frac{\partial}{\partial t} = -i\omega$

So in terms of these operations eqn (2) reduces to

$$\left.\begin{aligned} \vec{K}\cdot\vec{E} &= 0 \quad &\text{— (a)} \\ \vec{K}\cdot\vec{H} &= 0 \quad &\text{— (b)} \\ -\vec{K}\times\vec{H} &= \omega\epsilon\vec{E} \quad &\text{— (c)} \\ \vec{K}\times\vec{E} &= \omega\mu\vec{H} \quad &\text{— (d)} \end{aligned}\right\} \text{— — (9)} \qquad \left[\vec{K} = k\vec{n} = \frac{2\pi}{\lambda}\vec{n} = \frac{2\pi f}{v}\vec{n} = \frac{\omega}{v}\vec{n}.\right]$$

From this form of Maxwell's equation it is self evident that in a plane electromagnetic wave propagating through isotropic dielectric

(i) The vectors $\vec{E}$, $\vec{H}$ and $\vec{K}$ are orthogonal ie the electromagnetic wave is transverse in nature and in it the electric and magnetic vectors are mutually orthogonal. This is because

according to 9(a) $\quad \vec{E} \perp \vec{K}$

according to 9(b) $\quad \vec{H}$ is $\perp \vec{K}$

according to 9(c) $\quad \vec{E}$ is $\perp$ to both $\vec{K}$ and $\vec{H}$

according to 9(d) $\quad \vec{H}$ is $\perp$ to both $\vec{K}$ and $\vec{E}$.

(ii) The vectors $\vec{E}$ and $\vec{H}$ are in phase and their magnitudes are related to each other by relation

$$\left|\frac{\vec{E}}{\vec{H}}\right| = \frac{E_0}{H_0} = \sqrt{\left(\frac{\mu_r}{\epsilon_r}\right)} Z_0 = Z$$

where Z is called impedence of the medium

This is because according to eqn 9(d)

$$\vec{H} = \frac{\vec{K}\times\vec{E}}{\mu\omega} = \frac{k}{\omega\mu}(\vec{n}\times\vec{E}) = \frac{1}{\mu v}(\vec{n}\times\vec{E}) \quad \left(\text{as } K = \frac{\omega}{v}\right)$$

$$\text{or } \vec{H} = \frac{\sqrt{\mu\epsilon}}{\mu}(\vec{n}\times\vec{E}) = \sqrt{\frac{\epsilon}{\mu}}\,\vec{n}\times\vec{E} = \frac{\vec{n}\times\vec{E}}{Z} \quad \left[\text{as } v = \frac{1}{\sqrt{\mu\epsilon}}\right)$$

with $Z = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}} = \sqrt{\frac{\mu_r}{\epsilon_r}}\, Z_0$ (as $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$).

and $\frac{\mu_r Z_0}{n}$ (as $n = \sqrt{\mu_r \epsilon_r}$) $\left[\because \sqrt{\frac{\mu_r}{\epsilon_r}} = \sqrt{\frac{\mu_r \mu_r}{\mu_r \times \epsilon_r}} = \frac{\mu_r}{\sqrt{\mu_r \epsilon_r}} = \frac{\mu_r}{n}\right.$

or $Z = \frac{\mu_r Z_0}{n}$

or $\left|\frac{\vec{E}}{\vec{H}}\right| = \left|\frac{E_0}{H_0}\right| = Z =$ real quantity. — (10)

(iii) The direction of flow of energy in the direction in which the wave propagates and the Poynting vector is (n/μ_r) times of the poynting vector if the same wave propagates through free space

It is because $\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \frac{(\vec{n} \times \vec{E})}{Z}$

ie $\vec{S} = \frac{1}{Z}\left[\vec{n}(\vec{E}\cdot\vec{E}) - \vec{E}(\vec{E}\cdot\vec{n})\right]$

ie $\vec{S} = \frac{1}{Z} E^2 \vec{n}$ [as $\vec{E}\cdot\vec{n} = 0$ because $\vec{E}$ is $\perp$ to $\vec{n}$]

$$\vec{S} = \frac{E^2 \vec{n}}{Z} = \frac{E^2 \vec{n}\, n}{\mu_r Z_0} \quad \left(\text{as } Z = \frac{\mu_r Z_0}{n} \text{ and } \frac{1}{Z} = \frac{n}{\mu_r Z_0}\right)$$

$$\text{or } \vec{S} = \frac{E^2 \vec{n}\, n}{\mu_r} \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{E^2 \vec{n}\, n\, \epsilon_0 c}{\mu_r}$$

$$\vec{S} = \frac{n}{\mu_r} (\epsilon_0 c E^2) \vec{n}$$

$$\text{ie } \langle \vec{S} \rangle = \frac{1}{Z} E_{rms}^2 \vec{n} = \frac{n}{\mu_r} [\epsilon_0 c E_{rms}^2] \vec{n} \quad \text{—(11)}$$

(iv) The electromagnetic energy density is equal to the magneto-static energy density and the total energy density is ϵ_r times of the energy density if the same wave propagates through free space.

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\epsilon E^2}{\frac{1}{2}\mu H^2} = \frac{\epsilon}{\mu}\left(\frac{E}{H}\right)^2 = \frac{\epsilon}{\mu} Z^2 = \frac{\epsilon}{\mu}\frac{\mu}{\epsilon} = 1$$

$$\left(\text{as } \vec{H} = \frac{|\vec{E}|}{Z}\right)$$

$$\text{and } u = u_e + u_m = 2u_e = \epsilon E^2 = \epsilon_r (\epsilon_0 E^2)$$

$$\text{Further } \frac{\langle \vec{S} \rangle}{\langle u \rangle} = \frac{\frac{n}{\mu_r}[\epsilon_0 c E_{rms}^2]\vec{n}}{[\epsilon_r \epsilon_0 E_{rms}^2]} = \frac{nc}{\mu_r \epsilon_r} \vec{n}$$

$$\text{ie } \langle \vec{S} \rangle = \frac{nc}{n^2} \langle u \rangle \vec{n}$$

$$\text{ie } \langle \vec{S} \rangle = \frac{c}{n} \langle u \rangle \vec{n}$$

$$\text{ie } \langle \vec{S} \rangle = v \langle u \rangle \vec{n} \quad \left[\text{as } n = \sqrt{\mu_r \epsilon_r} \text{ and } \frac{c}{v} = n \Rightarrow \frac{c}{n} = v\right]$$

ie electromagnetic energy is transmitted with the same velocity with which the fields do.

PROPAGATION OF PLANE ELECTROMAGNETIC WAVES IN A CONDUCTING MEDIUM.

We Know the Maxwell's field equations are given by

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \text{and} \quad \vec{\nabla}\times\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{aligned}\right\} \quad \begin{aligned} \text{with} \quad \vec{D} &= \epsilon\vec{E} \\ \vec{B} &= \mu\vec{H} \\ \text{and} \quad \vec{J} &= \sigma\vec{E} \end{aligned}$$

We have considered a linear, homogeneous and conducting medium.
In the medium
ϵ = permittivity, μ = permeability
σ = Conductivity and the medium is charge free ie $\rho = 0$ and also external current free such that the currents existing in the medium are induced by the electromagnetic wave itself.
Thus $\vec{J} = \sigma\vec{E}$.
Any charge distribution within the conductor dies out quickly. The charge moves to the surface making $\rho = 0$ inside.

Thus the Maxwell's equations become

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0 && -(a) \\ \vec{\nabla}\cdot\vec{H} &= 0 && -(b) \\ \vec{\nabla}\times\vec{E} &= -\mu\frac{\partial \vec{H}}{\partial t} && -(c) \\ \vec{\nabla}\times\vec{H} &= \sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t} && -(d) \end{aligned}\right\} -(1)$$

Taking the curl of eqn (1c) and using eqn (1d)

$$\therefore \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}) = -\sigma\mu\frac{\partial\vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}$$

Now using eqn (1a) i.e. $\vec{\nabla}\cdot\vec{E}=0$

$$\nabla^2\vec{E} - \sigma\mu\frac{\partial\vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0 \quad \text{— (2)}$$

Similarly taking the curl of eqn (1d) and using (1c)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \sigma(\vec{\nabla}\times\vec{E}) + \epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

or $$\vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H} = -\sigma\mu\frac{\partial\vec{H}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

as using eqn (1b) i.e. $\vec{\nabla}\cdot\vec{H}=0$

$$\nabla^2\vec{H} - \sigma\mu\frac{\partial\vec{H}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} = 0 \quad \text{— (3)}$$

Eqns (2) and (3) are called 'equations of telegraphy'.

The plane wave solutions of eqn (2) and (3) are

$$\vec{E}(\vec{r},t) = \vec{E_0}\, e^{-i(\omega t - \vec{k}\cdot\vec{r})} \quad \text{and} \quad \vec{H}(\vec{r},t) = \vec{H_0}\, e^{-i(\omega t - \vec{k}\cdot\vec{r})} \quad \text{— (4)}$$

Where $\vec{E_0}$ and $\vec{H_0}$ are complex amplitudes which are constants in space time,

$\vec{K} = K\hat{n}$ is the propagation vector

Substituting the solution (4) in eqn (2) ~~and (3)~~ we have using the fact

$$\nabla^2\vec{E} = -K^2\vec{E}, \quad \frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E} \text{ and } \frac{\partial^2\vec{E}}{\partial t^2} = -\omega^2\vec{E}$$

$$\left[-K^2 + i\sigma\mu\omega + \omega^2\epsilon\mu\right]\vec{E} = 0$$

or For non-zero solution $$\left[-K^2 + i\sigma\mu\omega + \omega^2\epsilon\mu\right] = 0$$

$$K^2 = \omega^2 \epsilon \mu + i\omega\sigma\mu \quad — (5a)$$

K is thus a Complex quantity here

Let $K = \alpha + i\beta$ — (5,b) Then

$$K^2 = \alpha^2 - \beta^2 + i2\alpha\beta \quad — (6)$$

Comparing equations (5a) and (6)

$$\alpha^2 - \beta^2 = \omega^2 \epsilon \mu \quad \text{and} \quad 2\alpha\beta = \omega\sigma\mu$$

Now let us find the values of α & β. by solving the above two equations

$$(\alpha^2+\beta^2)^2 = (\alpha^2-\beta^2)^2 + 4\alpha^2\beta^2$$

$$(\alpha^2-\beta^2)^2 = (\omega^2\epsilon\mu)^2 \quad \text{and} \quad 4\alpha^2\beta^2 = (2\alpha\beta)^2 = (\omega\sigma\mu)^2$$

$$\therefore (\alpha^2+\beta^2)^2 = (\omega^2\epsilon\mu)^2 + (\omega\sigma\mu)^2$$

$$= (\omega^2\epsilon\mu)^2 \left[1 + \left(\frac{\omega\sigma\mu}{\omega^2\epsilon\mu}\right)^2\right]$$

$$(\alpha^2+\beta^2)^2 = (\omega^2\epsilon\mu)^2 \left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]$$

or $$\alpha^2+\beta^2 = \omega^2\epsilon\mu \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}$$

and $$\alpha^2 - \beta^2 = \omega^2\epsilon\mu \quad \therefore 2\alpha^2 = \omega^2\epsilon\mu \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]$$

or $$\alpha^2 = \omega^2 \frac{\epsilon\mu}{2} \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]$$

or $$\alpha = \pm \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{1/2}$$

Now as in the limit $\sigma \to 0$, $\mu \to \mu_0$; $\epsilon \to \epsilon_0$ and so eqn (5a) and (6) (5b) reduce to $k^2 = \omega^2 \epsilon_0 \mu_0$ and $k = \alpha$ so that $\alpha \Rightarrow \omega\sqrt{\epsilon_0 \mu_0}$

This in turn implies that correct value of α is given by

$$\alpha = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} + 1 \right]^{1/2} \quad — (7)$$

Similarly by solving we get

$$\beta = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} - 1 \right]^{1/2} \quad — (8)$$

Now in terms of α & β the field vectors $\vec{E}$ and $\vec{H}$ take the form

$$\vec{E} = \vec{E}_0 \, e^{-i\{\omega t - (\alpha + i\beta)\hat{n}\cdot\vec{r}\}}$$

$$\vec{E} = \vec{E}_0 \, e^{-\beta \hat{n}\cdot\vec{r}} \, e^{-i(\omega t - \alpha \hat{n}\cdot\vec{r})} \quad — (9)$$

and

$$\vec{H} = \vec{H}_0 \, e^{-i\{\omega t - (\alpha + i\beta)\hat{n}\cdot\vec{r}\}}$$

$$\vec{H} = \vec{H}_0 \, e^{-\beta \hat{n}\cdot\vec{r}} \, e^{-i(\omega t - \alpha \hat{n}\cdot\vec{r})} \quad — (10)$$

The above equation (9) and (10) show that the field amplitudes are spatially attenuated due to the presence of term $e^{-\beta \hat{n}\cdot\vec{r}}$

The quantity β is a measure of attenuation and is known as absorption coefficient or attenuation constant. It depends on frequency ω and conductivity σ.

Case (I) For a non-conducting medium $\sigma = 0$, $\beta = 0$

So there is no attenuation

Case (II) For conducting medium $\frac{\sigma}{\omega\epsilon} \gg 1$ so that

α and β are approximately equal

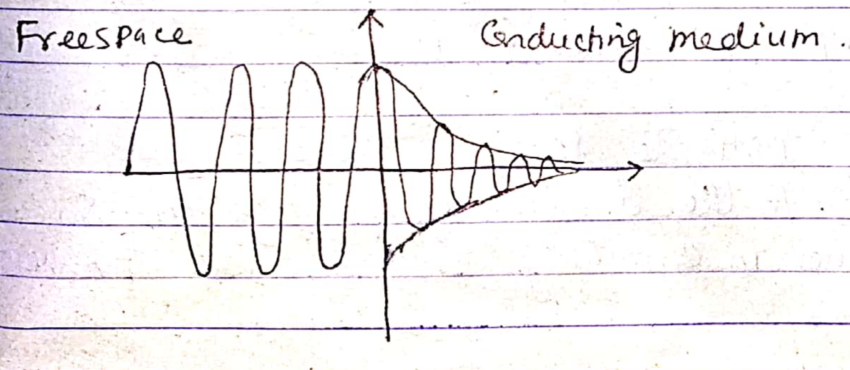
$$\alpha \approx \beta = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{\frac{\sigma}{\epsilon\omega}}\right)$$

$$\text{or } \beta = \sqrt{\frac{\omega^2\epsilon\mu\sigma}{2\omega\epsilon}} = \sqrt{\frac{\omega\mu\sigma}{2}} \quad \text{--- (11)}$$

The quantity $\frac{1}{\beta}$ measures the depth at which electromagnetic wave entering a conductor is attenuated to $\frac{1}{e}$ = 0.369 of its initial amplitude at the surface. It is known as skin depth or penetration depth into a conducting medium.

Thus, skin depth δ for a good conductor is

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma}} \quad \text{--- (12)}$$



Wavelength, propagation speed and the index of refraction:

The real part of K i.e. α determines the wavelength, the propagation speed of the wave and index of refraction of the conductor. Thus,

$$\lambda_c = \frac{2\pi}{\alpha}, \quad v = \frac{\omega}{\alpha} \quad \text{and} \quad n = \frac{c\alpha}{\omega}$$

where λ_c = wavelength, v = speed of propagation
n = refractive index

For a good conductor, $\alpha = \sqrt{\frac{\omega\sigma\mu}{2}}$ and hence

$$v = \frac{\omega}{\alpha} = \sqrt{\frac{\omega^2 \cdot 2}{\omega\sigma\mu}} = \sqrt{\frac{2\omega}{\sigma\mu}}$$

and refractive index $n = \frac{c\alpha}{\omega} = c\sqrt{\frac{\sigma\mu}{2\omega}}$

and the skin depth $\delta = \frac{\lambda_c}{2\pi}$ as $\lambda_c = \frac{2\pi}{\alpha}$

and $\frac{1}{\alpha} = \delta$ = skin depth

Relative directions of $\vec{E}$, $\vec{H}$ and $\vec{K}$:

From the form of solution of eqn of $\vec{E}$ and $\vec{H}$ given in eqn (4)

we have $\vec{\nabla} \rightarrow i\vec{K}$ and $\frac{\partial}{\partial t} \rightarrow -i\omega$

and substituting them in eqn (1a) & (1b)

we get $\vec{K}\cdot\vec{E} = 0$ and $\vec{K}\cdot\vec{H} = 0$

These equations indicate that $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation. So electromagnetic waves in conducting media are transverse in nature.

Also using $\vec{\nabla} \rightarrow i\vec{K}$ and $\frac{\partial}{\partial t} \rightarrow -i\omega$ in (1c) and (1d)

$$i\vec{K}\times\vec{E} = -\mu(-i\omega)\vec{H}$$

or $$\vec{K}\times\vec{E} = \omega\mu\vec{H} \quad — (13)$$

and $$i\vec{K}\times\vec{H} = \sigma\vec{E} + \epsilon(-i\omega)\vec{E}$$

or $$\vec{K}\times\vec{H} = -i\sigma\vec{E} - \epsilon\omega\vec{E}$$

or $$\vec{K}\times\vec{H} = -(\omega\epsilon + i\sigma)\vec{E} \quad — (14)$$

These equations imply that electromagnetic field vectors $\vec{E}$ and $\vec{H}$ are mutually perpendicular and are also perpendicular to the direction of propagation vector $\vec{k}$.

Relative Phase of $\vec{E}$ and $\vec{H}$.

From eqn (13), we have

$$\vec{H} = \frac{1}{\mu\omega}(\vec{k}\times\vec{E}) = \frac{K}{\mu\omega}(\hat{n}\times\vec{E}) = \frac{\alpha+i\beta}{\mu\omega}(\hat{n}\times\vec{E}) \quad \text{— (15)}$$

This implies that $\left|\frac{\vec{H}}{\vec{E}}\right| = \frac{H_0}{E_0} = \frac{\alpha+i\beta}{\mu\omega}$ = Complex quantity

ie $\vec{E}$ and $\vec{H}$ are not in phase in a conductor.

$$K = |K|\, e^{i\phi} \qquad \text{ie} \quad \alpha+i\beta = \sqrt{\alpha^2+\beta^2}\, e^{i\phi}$$

and $\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$ and using (15)

$$\vec{H} = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega}(\hat{n}\times\vec{E_0})\, e^{-i(\omega t - \vec{k}\cdot\vec{r} - \phi)} \quad \text{— (16)}$$

Where $\sqrt{\alpha^2+\beta^2} = \omega\sqrt{\epsilon\mu}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}$

Thus, $\vec{H}$ lags behind $\vec{E}$ in time by phase angle

$$\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right) *$$

$$* \quad \tan\phi = \frac{\beta}{\alpha} = \left[\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right]^{1/2} \quad \text{with } x = \frac{\sigma}{\omega\epsilon}$$

$$\therefore \tan 2\phi = \frac{2\tan\phi}{1-\tan^2\phi} = 2\left[\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right]^{1/2} \frac{1}{1-\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)}$$

$$\tan 2\phi = \cancel{2}\left[\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right]^{1/2} \times \frac{\sqrt{1+x^2}+1}{\cancel{2}}$$

or $\tan 2\phi = \left[\sqrt{1+x^2}-1\right]^{1/2} \cdot \left[\sqrt{1+x^2}+1\right]^{1/2}$ | i.e. $\tan^{-1} x = 2\phi$

or $\tan 2\phi = \left[1+x^2-1\right]^{1/2} = x$ | or $\phi = \frac{1}{2}\tan^{-1} x$

$= \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right)$

For good Conductors $\alpha \approx \beta$ and $\phi = 45°$. Therefore, the phase difference between the $\vec{E}$ and $\vec{H}$ fields in a perfect conductor is $45°$.

$$\left|\frac{\vec{H}}{\vec{E}}\right| = \frac{H_0}{E_0} = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} = \frac{\omega\sqrt{\epsilon\mu}}{\mu\omega}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}$$

$$\left|\frac{\vec{H}}{\vec{E}}\right| = \sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}.$$

For good Conductors; $\left|\frac{\vec{H}}{\vec{E}}\right| = \sqrt{\frac{\epsilon}{\mu}}\left[\sqrt{\frac{\sigma}{\omega\epsilon}}\right]$

as $\frac{\sigma}{\omega\epsilon} >> 1$

$$= \sqrt{\frac{\epsilon \cdot \sigma}{\mu\epsilon\omega}} = \sqrt{\frac{\sigma}{\mu\omega}}$$

or $$\left|\frac{\vec{H}}{\vec{E}}\right| = \sqrt{\frac{\sigma}{\mu\omega}}$$

Thus, in this case, $|\vec{H}| >> |\vec{E}|$ (as σ is very large for a good Conductor)

Which indicates that in a good Conducting medium the field energy is not equally shared between $\vec{E}$ and $\vec{H}$.

Poynting's Vector.

The time average of Poynting vector in this case is given by

$$\langle \vec{S} \rangle = \frac{1}{2}\text{ Real part of } (\vec{E}\times\vec{H}^*) = \frac{1}{2}\text{Re}(\vec{E}\times\vec{H}^*)$$

where $\vec{H}^*$ denotes Complex Conjugate of $\vec{H}$.

Now $$\vec{H} = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega}(\hat{n}\times\vec{E})\, e^{i\phi}$$

$$\therefore \quad \vec{H}^* = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} (\hat{n} \times \vec{E}^*) e^{-i\phi}$$

$$\therefore \quad \langle \vec{S} \rangle = \frac{1}{2} \cdot \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} \, Re \left[\vec{E} \times (\hat{n} \times \vec{E}^* \right] e^{-i\phi}$$

$$= \frac{\sqrt{\alpha^2+\beta^2}}{2\mu\omega} \, Re \left\{ \left[\hat{n} (\vec{E} \cdot \vec{E}^*) - \vec{E}^* (\vec{E} \cdot \hat{n}) \right] e^{-i\phi} \right\}$$

Now putting $\vec{E} \cdot \hat{n} = 0$ and using eqn (9) i.e
for $\vec{E} = \vec{E}_0 \, e^{-\beta \hat{n} \cdot \vec{r}} \, e^{-i(\omega t - \alpha \hat{n} \cdot \vec{r})}$

we have $\quad \vec{E} \cdot \vec{E}^* = E_0^2 \, e^{-2\beta \hat{n} \cdot \vec{r}}$

$$\langle \vec{S} \rangle = \frac{\sqrt{\alpha^2+\beta^2}}{2\mu\omega} E_0^2 e^{-2\beta \hat{n} \cdot \vec{r}} \cdot \cos\phi \; \hat{n} . \quad — (17)$$

For good conductors $\alpha = \beta \simeq \sqrt{\frac{\omega\sigma\mu}{2}}$ and $\phi = 45^\circ$. Then

$$\langle \vec{S} \rangle = \frac{\sqrt{\frac{\omega\sigma\mu}{2} + \frac{\omega\sigma\mu}{2}}}{2\mu\omega} E_0^2 e^{-2\beta \hat{n} \cdot \vec{r}} \cdot \frac{1}{\sqrt{2}} \hat{n}$$

$$= \frac{1}{2} \sqrt{\frac{\omega\sigma\mu}{2\mu^2\omega^2}} \, E_0^2 e^{-2\beta \hat{n} \cdot \vec{r}} \, \hat{n}$$

$$\langle \vec{S} \rangle = \frac{1}{2} \sqrt{\frac{\sigma}{2\mu\omega}} \, E_0^2 e^{-2\beta \hat{n} \cdot \vec{r}} \, \hat{n} \quad — (18)a$$

Thus, energy flow is along the direction of propagation of wave and is damped exponentially.

$$\langle \vec{S} \rangle = \sqrt{\frac{\sigma}{2\mu\omega}} \, E_{rms}^2 \, e^{-2\beta \hat{n} \cdot \vec{r}} \, \hat{n} \quad — (18b)$$

Energy density.

The average electrostatic energy density ~~of~~ is

$$\langle u_e \rangle = \frac{1}{2} \mathrm{Re}\, \frac{1}{2}(\vec{E}\cdot\vec{D}^*) = \frac{1}{4}\epsilon\, \mathrm{Re}(\vec{E}\cdot\vec{E}^*) = \frac{1}{4}\epsilon E_0^2 e^{-2\beta \hat{n}\cdot\vec{r}} \qquad (19)$$

and average magnetic energy density is

$$\langle u_m \rangle = \frac{1}{2} \mathrm{Re}\, \frac{1}{2}(\vec{B}\cdot\vec{H}^*) = \frac{1}{4}\mu\, \mathrm{Re}(\vec{H}\cdot\vec{H}^*) = \frac{1}{4}\mu H_0^2 e^{-2\beta \hat{n}\cdot\vec{r}} \qquad (20)$$

Thus $$\frac{\langle u_m \rangle}{\langle u_e \rangle} = \frac{\mu H_0^2}{\epsilon E_0^2} = \frac{\mu}{\epsilon}\cdot\frac{\epsilon}{\mu}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2}$$

Since $$\left|\frac{\vec{H}}{\vec{E}}\right| = \sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}$$

or $$\frac{\langle u_m \rangle}{\langle u_e \rangle} = \left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} \qquad (21)$$

For good conductors $\langle u_m \rangle >> \langle u_e \rangle$ i.e the field energy is almost entirely magnetic in nature.

The total average energy density is

$$\langle u \rangle = \langle u_e \rangle + \langle u_m \rangle = \frac{1}{4} e^{-2\beta \hat{n}\cdot\vec{r}}\left[\epsilon E_0^2 + \mu H_0^2\right] \qquad (22)$$

Using (21), we can write

$$\cancel{\langle u \rangle = \frac{1}{4} e^{-2\beta \hat{n}\cdot\vec{r}}\, \epsilon E_0^2 \left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2}}$$

$$\langle u_e \rangle + \langle u_e \rangle \left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2}$$

or $\langle u \rangle = \langle u_e \rangle \left[1 + \left\{ 1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2 \right\}^{1/2} \right]$

$$\langle u \rangle = \frac{1}{4} \epsilon E_0^2 e^{-2\beta \hat{n} \cdot \vec{r}} \left[1 + \left\{ 1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2 \right\}^{1/2} \right] \qquad (23a)$$

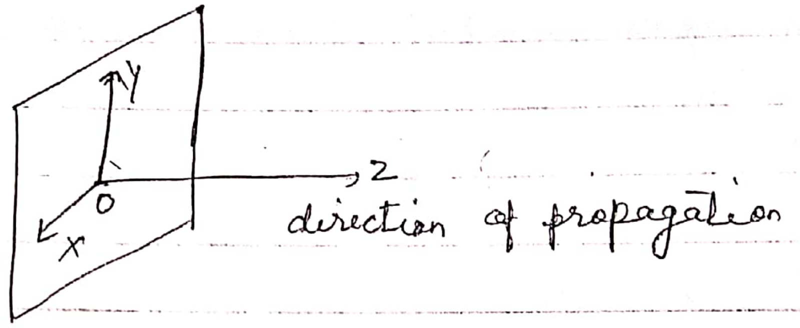
From eq (19) and (20) and (21) it is obvious that in a conducting medium the electrostatic and magnetostatic energy densities are different; the magnetostatic energy density being greater than electrostatic energy density.

$$\langle u \rangle = \frac{1}{2} \epsilon E_{rms}^2 e^{-2\beta \hat{n} \cdot r} \left[1 + \left\{ 1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2 \right\}^{1/2} \right] \qquad (23b)$$

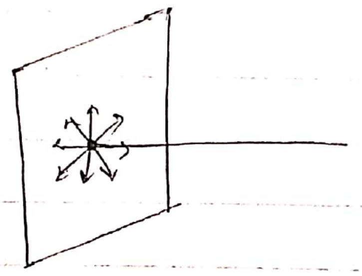
Polarisation of light :-

Ordinary light or unpolarised light :-

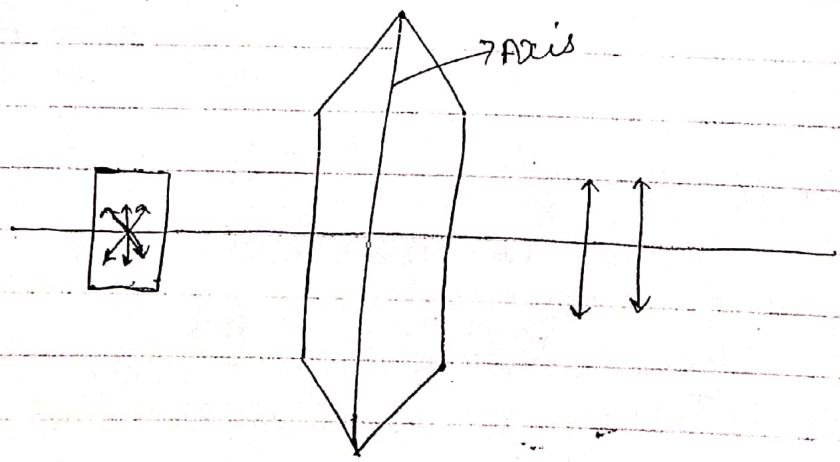
Electromagnetic wave in which electric & magnetic vectors vibrate in a direction ⊥r to direction of propagation of light



Ordinary or unpolarised light consists of symmetric vibrations in all directions in a plane ⊥r to direction of wave propagation.



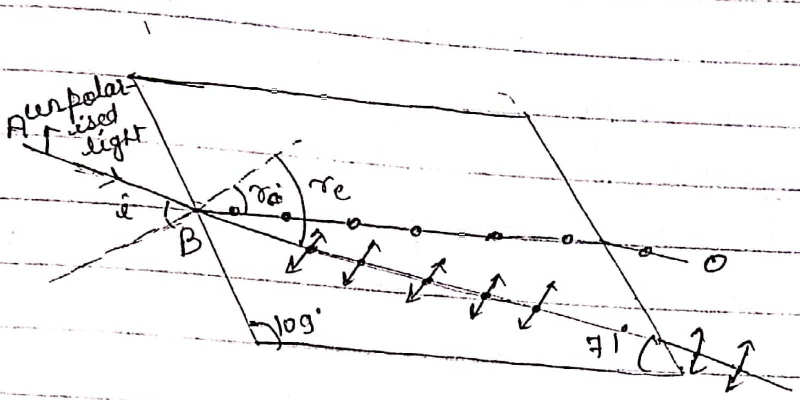
Tourmaline crystal



The phenomenon of setting light vibrations in a particular direction is called Polarisation of light.

Double refraction :-

When a ray of unpolarised light is incident on a calcite or quartz crystal it is split ~~a~~ up into two refracted rays. This phenomenon is called double refraction.



One of the two refracted rays is found to obey the laws of refraction and is called ordinary ray (O-ray) while other do not obeys those laws is called extraordinary rays (e-ray). Both of these rays are plane polarised whose vibrations are $\perp r$ to each other.

Let the ray AB is incident on crystal at an angle i and undergoes double refraction. The refractive index for ordinary rays is

$$\mu_o = \frac{\sin i}{\sin r_o}$$

and refractive index for extra ordinary rays is

$$\mu_e = \frac{\sin i}{\sin r_e}$$

For calcite crystal

$$r_o < r_e$$

$$\& \text{ so } \mu_o > \mu_e$$

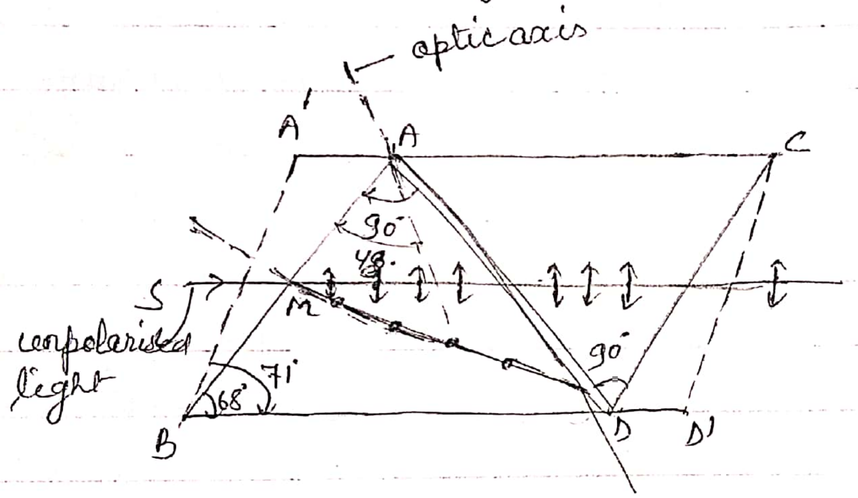
Hence in calcite crystal velocity of o-ray is less than e-ray. This type of crystal is called uniaxial (-)ve crystal.

For quartz $\mu_e > \mu_o$

& so velocity of o-ray through quartz ray is greater than E-ray. This type of crystal is called uniaxial (+)ve crystal.

Nicol prism :-

It is an optical device made from a calcite crystal for producing and analysing plane polarised light



It consist of a calcite crystal A B whose length is about three ~~ty~~ times of its width. It end faces A B and C D are cuts ~~of~~ such that angles of principal section becomes 68° and 112° instead of 71° & 109°

The crystal is cut diagonally along plane A D ⊥r to the both the principal section and end faces A B & C D. The two cut surfaces are grounded and polish optically flat. They are cemented together by canada balsam.

Action:-

Refractive index for calcite = 1·66 for o-ray
" " " canada balsam = 1·55
" " " calcite = 1·49 for E-ray

When a ray SM of unpolarised light ||r to BD is incident on face AB, it is divided into two components:- O-ray & E-ray

The O-ray is total internally ~~refracted~~ reflected at balsam layer provided angles of incident exceeds 69° (critical angle) while E-ray is transmitted, since E-ray is polarised ~~the~~ the light emerging from Nicol prism is plane polarised with vibration ||r to principal system.

Limitation:-

Nicol prism works only when the incident beam is slightly convergent and divergent. If the incident ray make much smaller angle than angle SMB the ordinary ray will strike the balsam layer ~~at~~ at angle less than critical angle (69°) & hence it will be transmitted.

Uses:-

The Nicol prism is used both for polaraliser or analysier.

When an unpolarised light is incident on Nicol prism the ray emerging from prism is plane polarised and has plane vibration ||r to prism section

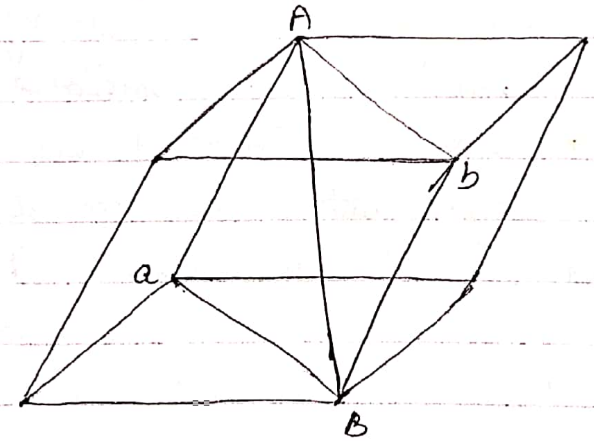
If this ray falls on 2nd prism whose principal section is ∥r to 1st the intensity of emergent light will be max^n.

Now, If the 2nd Nicol prism is roted, the intensity will go on decreasing and finally becomes zero. when its principal section ⊥er to 1st. The 1st Nicol prism is called polariser while 2nd analysier.

<u>Optic axis</u> :-

For an uniaxial crystal, the direction along which E-ray is equal to that of O-ray is called optic axis.

Calcite ($CaCO_3$ or iceland spar) is ~~rhombo~~ rhombohedron in which 6 faces are ~~lp.~~ parallelogram each having angle 102° & 78° at two opposite ~~as~~ corners such as A & B three angles



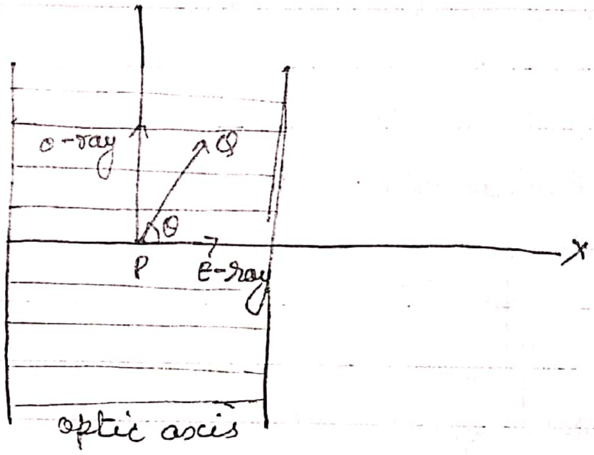
of the faces meeting their ~~axis~~ are obtuse (more than 90°) these corners are known as blunt corners. A line equally inclined to ~~8~~ 3-edges meeting at one of blunt corners A & B or any other line ∥r to ~~it each~~ it is direction of optic ascis ~~is a~~ of crystal. Thus optic axis is dir^n ~~in~~

in which velocity of O-ray & E-ray are equal.

Principal Plane or Section:-

A plane containing optic axis and ⊥r to face of crystal is called principal plane or section. In fig AaBb is principal plane for the top & bottom faces of crystal

Theory of polarised light in calcite crystal:-



Let plane polarised monochromatic light is incident normally on a calcite crystal cut with its faces ||r to optic axis. let the linear vibration in the incident light be along PQ making an angle θ with optic axis and has amplitude A.

On entering the crystal the amplitude of incident ~~ray~~ light split up into two components A cosθ along P and A sinθ along PO. The component A cosθ having vibration ||r to optic axis form E-ray and component A sinθ having vibration ⊥r to the optic axis

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form o-ray. These two will travel in same direction but with different velocity. In calcite and other (−) ve crystal e-ray travel faster than o-ray.

when the two rays are emerge out of the crystal a phase difference 'δ' introduce betn them.

The two emergent plane polarised E and O-waves are represented as

$$x = A\cos\theta \sin(\omega t + \delta) = a\sin(\omega t + \delta) \quad \text{--- (1)}$$

$$\& \; y = A\sin\theta \sin\omega t = b\sin\omega t \quad \text{--- (2)}$$

where $a = A\cos\theta$

$b = A\sin\theta$

From (2), $\sin\omega t = \frac{y}{b}$

$\& \; \cos\omega t = \sqrt{1 - \frac{y^2}{b^2}}$

Also from (1) $\frac{x}{a} = \sin\omega t \cdot \cos\delta + \cos\omega t \cdot \sin\delta$

or $\frac{x}{a} = \frac{y}{b}\cos\delta + \sqrt{1 - \frac{y^2}{b^2}}\, \sin\delta$

or $\frac{x}{a} - \frac{y}{b}\cos\delta = \sqrt{1 - \frac{y^2}{b^2}}\, \sin\delta$

or $\left(\frac{x}{a} - \frac{y}{b}\cos\delta\right)^2 = \left(1 - \frac{y^2}{b^2}\right)\sin^2\delta$

$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\delta + \frac{y^2}{b^2}\cos^2\delta = \sin^2\delta - \frac{y^2}{b^2}\sin^2\delta$

or $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\delta + \frac{y^2}{b^2} = \sin^2\delta \quad \text{--- (3)}$

This represents eq^n of an oblique ellipse and so the light emerging from crystal is elliptically polarised.

Case I :- If thickness of plate is such that $\delta = 0, 2\pi, 4\pi - - -$

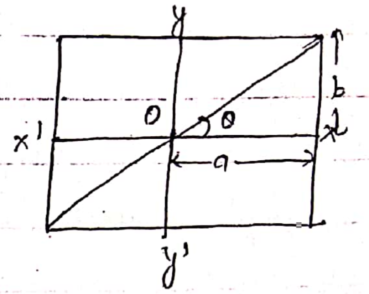
Then $\cos\delta = 1$ & $\sin\delta = 0$

Eqⁿ ③ becomes.

$$\frac{x^2}{a^2} - \frac{2xy}{ab} + \frac{y^2}{b^2} = 0$$

$$\left(\frac{y}{b} - \frac{x}{a}\right)^2 = 0$$

or $y = \frac{b}{a}x$



This represents a pair of coincident straight lines through origin having ~~been~~ (+)ve slope $\frac{b}{a}$.

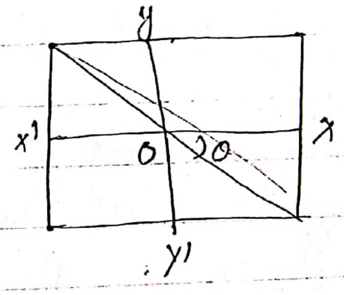
Case II :- If $\delta = \pi, 3\pi, 5\pi, - - -$

Then $\cos\delta = -1$ & $\sin\delta = 0$

Eqⁿ ③ ~~be~~ gives

$$\frac{x^2}{a^2} + \frac{2xy}{ab} + \frac{y^2}{b^2} = 0$$

or $y = -\frac{b}{a}x$



Thus again represents a pair of coincident straight lines but with (−)ve slope $-\frac{b}{a}$. is obtained. Hence emergent light is linearly polarised.

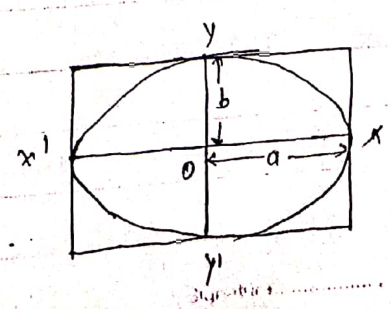
Case III - If $\delta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, - - -$

then $\cos\delta = 0$ & $\sin\delta = 1$

Eqⁿ ③ reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It represents an ellipse with its axes along x & y - axes.



and so light emerging from crystal is elliptically polarised.

~~In this case~~ If $\delta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots$ then

$$x = a \cos \omega t$$

$$\& \; y = b \sin \omega t$$

These eq^ns shows that ellipse is described by tip of light vector rotating counter clockwise. The Emergent light ~~is~~ is said to be left handed elliptically polarised.

~~If $\delta = \frac{\pi}{2}$,~~

But if $\delta = \frac{3\pi}{2}, \frac{7\pi}{2}, \dots$ etc

$$x = -a \cos \omega t$$

$$\& \; b = b \sin \omega t$$

In this case light is said to be right handed elliptically polarised.

However, if in this case $\theta = \omega t = 45^\circ$ then $a = b$

so, $x^2 + y^2 = a^2$

In this case emergent light is circularly polarised

Retarding plates :-

A retarding plate is a plate cut from a refracting crystal by section $\parallel$ to optic axis and employed to introduce a required phase difference bet^n O-ray & E-ray on transmission normally through it.

let, t = thickness of plate in the direction of propagation

μ_o = Refractive index of o - ray
μ_e = " " " E- ray
$\mu_o t$ = optical path for o - ray
$\mu_e t$ = " " " E- ray

For negative crystal such as calcite
$\mu_o > \mu_e$

& so path-difference bet^n o - ray & E - ray
$= \mu_o t - \mu_e t = (\mu_o - \mu_e) t$

So, phase difference $\delta = \frac{2\pi}{\lambda}(\mu_o - \mu_e) t$

For (+) ve crystal as quartz i.e. $\mu_e > \mu_o$ & so
$\delta = \frac{2\pi}{\lambda}(\mu_e - \mu_o) t$

There are two types of retarding plates :-
a) Quarter wave plate ($\lambda/4$ plate)
& b) Half wave plate ($\lambda/2$ plate)

a) Quarter wave plate :- A doublly refracting plate having thickness such as to produce a path difference $\lambda/4$ & phase difference $\pi/2$ bet^n the o- ray & E - ray is called Quarter wave plate.

For (-)ve crystal,
$$(\mu_o - \mu_e) t = \frac{\lambda}{4}$$

or $$\boxed{t = \frac{\lambda}{4(\mu_o - \mu_e)}}$$

Similarly for (+) ve crystal,
$$t = \frac{\lambda}{4(\mu_e - \mu_o)}$$

Hence a plate of thickness given by

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this eq^n will serve as quarter wave plate for a given wave lenth λ

USE :-

Quarter wave plate is used for producing circularly & elliptically polarised light.

b) Half wave plate :- A doublly refracting plate having thickness such as to produce a path-difference $\lambda/2$ or phase difference π bet^n the O-ray & E-ray is called half wave plate or $\lambda/2$ plate.

For (-)ve crystal, such as calcite

$$(\mu_o - \mu_e)\, t = \frac{\lambda}{2}$$

USE :-

Such a plate is used in polarimeter.

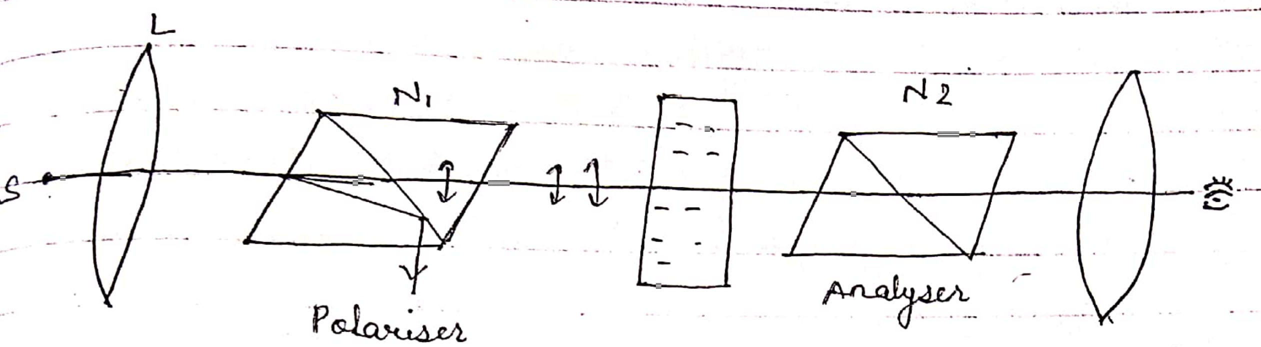
Q. what do you understand by optical rotation or optical activity give its theory.

When plane polarised light passes through certain substances, the plane of polarisation of light is rotated about the direction of propagation of light through a certain angle. This phenomenon is called optical rotation or rotatory polarisation.

Substances which show this phenomenon are called optically active substances. Those which rotates the plane of polarisation is called right handed or Dextrogyrate and those which rotate anti clockwise is called left handed or laevogyrate. many liquids or organic substances in sol^n

are found to be optically active.

Demonstration :-



When plane polarised light emerging from N_1 is examine through another analyser N_2 it is completely cut off when the principal section of N_2 is ⊥r to that of N_2. But if a quartz plate C cut with optic axis ⊥er to its face the place bet^n N_1 & N_2 some light is observed to pass through N_2. The light is again completely cut of if N_2 is rotated through certain angle. This shows that light emerging from quartz plate is still plane polarised but its plane of polarisation is rotated by ~~an~~ certain angle.

Laws of rotation of plane of polarisation

1) The angle of rotation of plane of polarisation for a given wave length is directly proportional to the length of optically active substance traversed.

2) For sol^n & vapours the angle of rotation is porportional to the concentration of the sol^n or vapour.

3) The rotation produced by a no. of optically active substance is equal to the algebric sum of the individual rotation. The anti clock wise or ~~col~~ clock wise rotation taken with opposite sign.

4) The angle of rotation is approximately inversly porportional to square of wave-length.

Fresnel's explanation of Rotatary Polarisation :-

Fresnel's explanation of optical rotation is based on the fact that a linear vibration may be described as the resultant of two opposite circular motion of same frequency. He made the following assumption :-

1) The incident polarised light on entering the substance is broken up into two circularly polarise waves, one clockwise & other anti clockwise.

2) In optically inactive substance the two waves travel with same velocity but in an optically active substance they traveled with different velocity. Hence a phase difference develop betn them.

3) on emergences the two circular components re-combine to form plane polarised light whose plane of polarisation is rotated w.r. to that of incident light by an angle depending upon the phase difference betn them

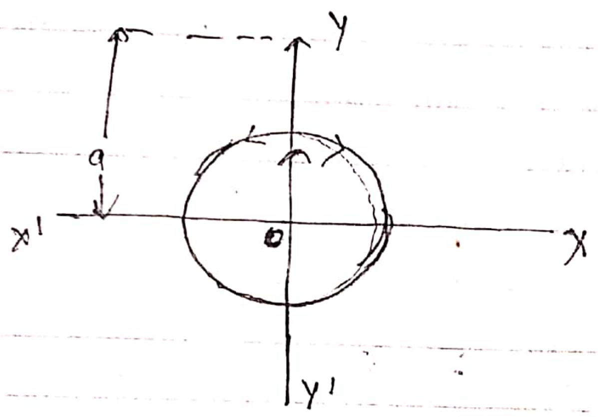
let, the vibrations in the incident light is given by

$$y = a\cos\omega t \quad \text{——— (1)}$$

This vibration on entering the crystal are broken up into two equal & opposite circular motion represented by

$$\left.\begin{aligned} x_1 &= \frac{a}{2}\sin\omega t \\ \& \; y_1 &= \frac{a}{2}\cos\omega t \end{aligned}\right\}\text{clockwise circular motion}$$

$$\text{and}\quad \left.\begin{aligned} x_2 &= -\frac{a}{2}\sin\omega t \\ y_2 &= \frac{a}{2}\cos\omega t \end{aligned}\right\}\text{anticlockwise circular motion.}$$



These components are propagated through the substance with different velocity & on emergence a phase difference δ is introduced.

The emergent circular components can be represented by

$$x_1 = \frac{a}{2}\sin(\omega t + \delta)$$

$$y_1 = \frac{a}{2}\cos(\omega t + \delta)$$

$$\& \quad x_2 = -\frac{a}{2}\sin\omega t$$

$$y_2 = \frac{a}{2}\cos\omega t$$

Resultant displacement along x-axis is

$$x = x_1 + x_2 = \frac{a}{2}\left[\sin(\omega t + \delta) - \sin \omega t\right]$$

$$= a \cos\left(\omega t + \frac{\delta}{2}\right) \cdot \sin\frac{\delta}{2}$$

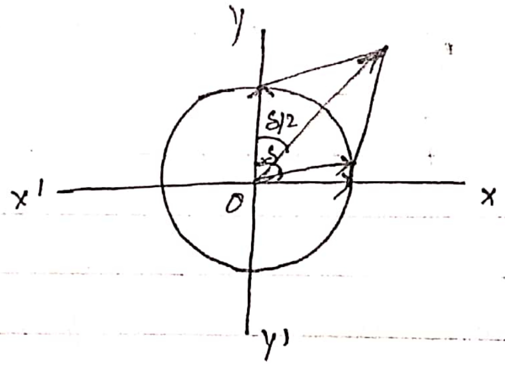
and along y-direction is

$$y = y_1 + y_2 = \frac{a}{2}\left[\cos(\omega t + \delta) + \cos \omega t\right]$$

$$= a \cos\left(\omega t + \frac{\delta}{2}\right) \cos\frac{\delta}{2}$$

Also $\frac{x}{y} = \tan\frac{\delta}{2}$

This represent a st. line and so the light emerging from plate is plane polarised with vibrations inclined at angle $\delta/2$ with y-axis



i.e. to the vibration of incident wave.

If μ_A & μ_C are refractive indexes of quartz in the direction of optic axis for anticlock wise & clock wise circularly polarised light respec- & 'd' is thickness of crystal plate. ~~The phas~~

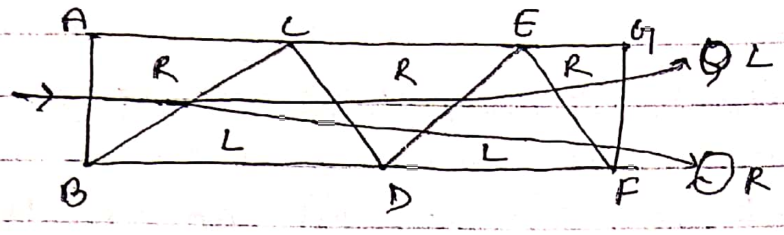
The phase difference δ is given by

$$\delta = \frac{2\pi}{\lambda}(\mu_A - \mu_C)\, d$$

Hence rotation of plane of polarisation

$$= \frac{\delta}{2} = \frac{\pi(\mu_A - \mu_C)\, d}{\lambda}$$

Experimental verification of Fresnel's Theory :-



ABFG = a rectangular block made of alternate prisms of right handed & left handed quartz, all having their optic axis ⊥er to end faces AB & FG.

To verify Fresnel's theory, a beam of plane polarised light is made to ~~form~~ fall normally & rectangular block. This light will break up into two opposite circularly polarised wave which traveled the 1^{st} prism with different velocity but in same direction. In 2^{nd} prism let R-waves which was faster in the 1^{st} prism become the slower in the 2^{nd} while L-wave will be faster. Hence the 2^{nd} prism is a denser medium for R-wave and rarer medium for L-wave. Hence in the 2^{nd} prism R-wave bends towards base and L-wave away from the base. At the 2^{nd} boundary CD the velocities are again interchanged so that R-wave bends away from the base and L-wave towards the base. The net result is that angular separation increases at each boundary.

when the two emerging waves are analyse by $\lambda/4$ plate and a nicol prism, they are found to be circularly polarised in a opposite direction which verifies Fresnel's theory.